Tâtonnement, Approach to Equilibrium and Excess Volatility in Firm Networks

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We study the conditions under which input-output networks can dynamically attain competitive equilibrium, where markets clear and profits are zero. We endow a classical firm network model with simple dynamical rules that reduce supply/demand imbalances and excess profits. We show that the time needed to reach equilibrium diverges as the system approaches an instability point beyond which the Hawkins-Simons condition is violated and competitive equilibrium is no longer realisable. We argue that such slow dynamics is a source of excess volatility, through accumulation and amplification of exogenous shocks. Factoring in essential physical constraints, such as causality or inventory management, we propose a dynamically consistent model that displays a rich variety of phenomena. Competitive equilibrium can only be reached after some time and within some region of parameter space, outside of which one observes periodic and chaotic phases, reminiscent of real business cycles. This suggests an alternative explanation of the excess volatility that is of purely endogenous nature. Other regimes include deflationary equilibria and intermittent crises characterised by bursts of inflation. Our model can be calibrated using highly disaggregated data on individual firms and prices, and may provide a powerful tool to describe out-of-equilibrium economies.

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CONTENTS

I. INTRODUCTION

Classical macroeconomic models picture the world as a succession of equilibria where markets clear perfectly and firms make no profit. Each equilibrium is characterized by a different level of productivity or household preferences, themselves driven by exogenous "shocks". Drawing an analogy from physics, one may call such an approach "adiabatic" – i.e. the time needed for the system to reach equilibrium is much shorter than the time over which the environment changes, so that one can compute the properties of the system as if the environment was static. The time evolution of the economy is then slaved to the time evolution of the exogenous parameters.

There are however many reasons to believe that the economy is permanently out-of-equilibrium. One of these reasons is the "small shock, large business cycle" paradox: aggregate fluctuations seem much too large to be explained by exogenous shocks alone [1–3]. Some endogenous dynamics, intrinsic to economic systems, appear to be at play, like in financial markets (see e.g. [4], ch. 20, for a recent review). From a conceptual point of view, economic equilibrium requires so much cooperation between rational, forward looking agents, that the only way such equilibrium can plausibly be achieved is through some kind of learning, or *tâtonnement*, that inevitably takes some time to complete.[1]

If this time is comparable to, or longer than, the evolution time of technology, or of any other type of events (political, social, geo-political, sanitary, etc.) that do affect the economy, then the adiabatic hypothesis is doomed to fail, and calls for a richer modelling framework where dynamics is an integral part of the description. We do not only need to describe the equilibrium state, but also the path to equilibrium.[2] We might realize that in some cases it is in fact never reached – opening the possibility of purely endogenous macro-economic fluctuations.

There is of course a large literature on out-of-equilibrium macroeconomics, see e.g. [9–15]. Part of this literature is concerned with "disequilibrium", i.e. the impact of frictions and price or wage rigidities, that prevent the economy from reaching equilibrium, but with no particular focus on dynamical effects. Another strand of literature postulates "reduced form" differential equations that describe the coupled evolution of a set of aggregate variables (e.g. employment, wage and output in the Goodwin model [16, 17]). These low-dimensional dynamical equations can generate interesting phenomena, such as business cycles in the Goodwin model which is, *mutatis mutandis*, equivalent to the classic Lotka-Volterra (or predator-prey) model [18, 19]. Yet another direction is explored by Agent Based Models (ABM), where individual agents/firms make decisions based on plausible heuristic rules. ABM are explicitly dynamical models [20]: decision rules lead to actions (buy/sell, produce, update prices and wages, etc.) that move the economy one step forward in time (see e.g. [21–26]). Although these three different avenues of research have led to a considerable number of papers in the last decades, they are often spurned by mainstream macroeconomists who prefer "micro-founded" models where agents/firms are forward looking and optimize inter-temporal utility functions.

In this paper, we want to revisit these ideas within the framework of network economies, where firms interact through a supply/demand (or input/output) network. Such models have recently become popular as a way to generate excess

[1] This is true even for financial markets where transactions take place at the second time scale. In reality, a large amount of the supply/demand volume is latent and is only slowly revealed [4–6].
[2] In fact, if one delves into the history of the notion of economic equilibrium from Walras up to the Arrow-Debreu general equilibrium theory, it is striking to see that the focus has been mainly on the existence and the properties of an economic equilibrium. It is assumed that a mechanism exists that leads the economy towards that point, but it is not made explicit [7, 8].

aggregate volatility, as shocks may possibly propagate through the input-output network. However, the seminal papers of Long & Plosser [1], and of Acemoglu, Carvalho & collaborators [27] are studied within the "adiabatic" framework discussed above. Furthermore, these papers assume a Cobb-Douglas production function, which ensures that an equilibrium always exists, whatever the input-output network and independently of the productivities of the firms. But as shown in [28], for more general production functions (such as the Constant Elasticity of Substitution – CES – family) equilibrium ceases to exist when the average connectivity of the network is too large, firm productivities are too low, or markups are too large. In these cases, the description of a time evolving economy as a succession of static equilibria just does not make sense. In fact, Moran & Bouchaud [28] argue, in the spirit of a conjecture by Bak, Chen, Scheinkmann and Woodford [29], that economies may generically sit close to a point where equilibrium disappears.

One therefore needs to endow the model with plausible dynamical rules, which would allow one to follow the fate of the economy outside of the adiabatic regime, and in fact identify cases where equilibrium does exist mathematically but can never be reached. A step in this direction was proposed by Mandel et al. [30] and, independently, by Bonart et al. [31], where a dynamical Cobb-Douglas economy was considered, with plausible update rules for production and prices. Interestingly, the model considered in [31] leads to a phase transition between a region where equilibrium is reached (when firms slowly adapt to shocks) and a region where coordination breaks down (when firms adapt too aggressively) and where equilibrium is no longer dynamically accessible.[3] In the latter phase, endogenous volatility becomes dominant. But this model only goes half-way towards a full-fledged dynamical description, since market-clearing was imposed by *fiat* in [31], with no excess production or excess demand – leading to conceptual inconsistencies and, in fact, spurious instabilities.

In the present work, we attempt to provide a consistent framework to describe dynamical out-of-equilibrium effects in network economies. Our approach is a hybrid between classical economics thinking (where firms attempt to optimize profits in a competitive environment, and households optimize their utility function to balance consumption and labour) and Agent Based Models, where simplified behavioural assumptions allow one to specify the decision-making process of firms. Much to our surprise, we have found that in order to obtain well-behaved outcomes, extra care has to be devoted to treat all the decision steps in a strictly causal way (for example, goods must be produced before they are consumed) and to satisfy all inequalities (for example, consumption cannot exceed production plus inventories). Any attempt to write down "reasonable" dynamical equations that violate these constraints consistently lead to spurious instabilities. We in fact consider this as a blessing: physical constraints provide a discriminant straight-jacket for modelling. We propose a minimal parametrisation of the heuristic rules used by firms to update production, prices and wages, which already leads to a surprisingly rich phenomenology of the resulting economy.

In a sense, our model can be seen as a multidimensional, discrete time version of the reduced form differential equations *à la* Goodwin [16] and followers. The main difference is that we describe the dynamics of the economy at a highly disaggregated level (that of firms), which is an important aspect in view of the amount of micro-data now available to calibrate such a model. In view of the diversity of phenomena that can take place within our framework, we are quite confident that the model is flexible enough to account for many empirical facts. However, in the current era of "big data", some extensions of the model may be worthwhile investigating – each extension bringing one or several new parameters that need to be calibrated. In particular, some of our behavioural assumptions may appear too primitive and could be enriched, as we discuss in section IV C.

The most important generalisation, in our opinion, will be to include debt, interest rates and bankruptcies in the model. In particular, the way the network "rewires" after the removal of a bankrupt firm, with the possibility of cascading defaults, is clearly one of the most interesting aspects of firm network models when it comes to understanding economic crises and, in fact, the very motivation for studying network models.

The manuscript is organized as follows. In section II, we set up the stage of firm networks at equilibrium. Section III presents a simple heuristic for an out-of-equilibrium dynamical model of interacting firms. We show that reaching equilibrium might take an infinite amount of time (therefore jeopardizing the adiabatic hypothesis) and that the dynamics displays excess volatility when the economy sits close to an instability. In section IV, we present a fully consistent extension of the model of section III which incorporates natural constraints which were overlooked such as causality or shortages. We propose a preliminary numerical study of this extension in section V where we highlight and discuss the existence of other interesting dynamical regimes besides competitive equilibrium. We also provide several technical appendices for completeness. In Appendix A, we detail the derivation of competitive equilibrium equations in the most general setting of production function. Appendix B shows the computation of the relaxation time of

[3] A similar phenomenology is reported in Ref. [30], where it is stated that *depending on the stringency of the financial constraints the model can settle in two very different regimes: one characterized by equilibrium, the other by disequilibrium and financial fragility.*

the naive model which relies on Appendix C that compiles necessary intermediate results on the stability matrix. In Appendix D we show that a marginally stable linear stochastic system creates excess volatility and we apply this result to a generic case of the naive model of III. Finally in Appendix E, we provide a pseudo-code for simulation of the fully consistent approach of section IV. The code itself is made available at: `https://yakari.polytechnique.fr/dash`.

II. FIRM NETWORKS AT EQUILIBRIUM

A. Network and production function

Following references [1, 27, 31], we model the economy as consisting of N firms that interact with one another and with a single representative household which provides labour and consumes goods. The economy is described by a "technology network", namely a directed graph where each node $i = 1, \dots, N$ represents a firm and where the link $j \to i$ exists if i uses the good produced by j for its own production. The node labelled $i = 0$ conventionally represents households. Each edge in the graph $i \to j$ carries a "weight" that is a measure of the number of j goods needed to make an unit of i. The production function gives the quantity of goods π_i produced by i as a function of input goods and labour (no capital at this stage) and the intrinsic, possibly time dependent, productivity of the firm z_i (i.e. its efficiency in converting a given amount of inputs into outputs). The standard CES production function writes [32]:

$$\pi_i = z_i \left(\sum_{j=0}^{N} a_{ij} \left(\frac{Q_{ij}}{J_{ij}} \right)^{-1/q} \right)^{-bq} := z_i \gamma_i, \tag{II.1}$$

where Q_{ij} is the amount of good j (or labour if $j = 0$) available to i, $J_{ij} \geq 0$ and $a_{ij} \geq 0$ link variables that measure the importance of good j in the production of i[4], and where we define γ_i as the level of production of firm i. The parameter b sets the return to scale: if all inputs and work hours are multiplied by a factor λ, then total output is multiplied by λ^b.

Finally, q is a parameter measuring the substitutability of inputs. For example, when $q \to 0^+$ we get the Leontief production function

$$\pi_i = z_i \left(\min_j \left(\frac{Q_{ij}}{J_{ij}} \right) \right)^b,$$

corresponding to the case where production falls to zero if a single input is missing. This represents a setting where firms only keep a small, very optimized portfolio of suppliers that does not allow for redundancy. If $q \to +\infty$, we get the Cobb-Douglas production function

$$\pi_i = z_i \left(\prod_j \left(\frac{Q_{ij}}{J_{ij}} \right)^{a_{ij}} \right)^b,$$

for which some amount of substitutability is present. Indeed, halving the quantity Q_{ik} of input k can be compensated by multiplying the input of ℓ by $2^{a_{ik}/a_{i\ell}}$.

Although our dynamical model applies to any production function, and is not restricted to the CES family specified above, we will illustrate our general arguments using the special case of a Leontief production function with constant return to scale ($b = 1$), as its simplicity allows for equilibrium conditions to be worked out explicitly.

B. Equilibrium conditions on prices and productions

Given the prices p_i of the goods and wage p_0, the profit $\mathscr{P}_i$ of firm i can be written as

$$\mathscr{P}_i = \sum_j Q_{ji} p_i - \sum_j Q_{ij} p_j, \tag{II.2}$$

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[4] The a_{ij} are normalized such that $\sum_j a_{ij} = 1$, $\forall i$.

where $Q_{0i} := C_i$ is the consumption of good i by the households. For a certain target production $\widehat{\pi}_i := z_i \widehat{\gamma}_i$, the optimal quantities $\widehat{Q}_{ij}$ that the firm needs to buy (including workforce) are obtained by minimising the second term in Eq. (II.2), corresponding to production costs, with the target production constraint. Within the CES framework, this leads to:

$$\widehat{Q}_{ik} = a_{ik}^{q\zeta} J_{ik}^{\zeta} \left(\sum_j a_{ij}^{q\zeta} J_{ij}^{\zeta} \left(\frac{p_j}{p_k} \right)^{\zeta} \right)^{q} \widehat{\gamma}_i^{1/b}, \tag{II.3}$$

with $\zeta = (1+q)^{-1}$. In the Leontief case with $b = 1$, this boils down to

$$\widehat{Q}_{ik} = J_{ik} \widehat{\gamma}_i. \tag{II.4}$$

Equilibrium prices and productions are then fully determined by assuming perfect competition, i.e. $\mathscr{P}_i = 0$ for all firms, and perfect market clearing, meaning that all of the production is consumed. This last condition can be written as

$$\pi_{\mathrm{eq},i} = C_{\mathrm{eq},i} + \sum_j Q_{\mathrm{eq},ji}, \tag{II.5}$$

where C_i is the households' demand for good i. For Leontief production functions with $b = 1$, the resulting equations are linear and read:

$$\boldsymbol{\mathcal{M}}\, \mathbf{p}_{\mathrm{eq}} = \mathbf{V} \tag{II.6a}$$
$$\boldsymbol{\mathcal{M}}^t\, \boldsymbol{\gamma}_{\mathrm{eq}} = \frac{\boldsymbol{\kappa}}{\mathbf{p}_{\mathrm{eq}}}, \tag{II.6b}$$

where $\boldsymbol{\mathcal{M}}$ is a matrix defined as $\boldsymbol{\mathcal{M}}_{ij} = z_i \delta_{ij} - J_{ij}$, $\mathbf{V}_i = J_{i0}$ is the workforce need of firm i and $\boldsymbol{\kappa}$ a positive vector describing final demand.[5] For more general production functions, the equations can be written down as well – see Appendix A – but we will not consider them in the present paper. The important features are:

- For Eqs. (II.6a, II.6b) to have positive solutions for prices and productions, $\boldsymbol{\mathcal{M}}$ must be a so-called M-matrix [28, 33, 34]. Owing to its particular shape, with non-negative terms on the diagonal and negative terms on the off-diagonal, this is equivalent to the spectrum of $\boldsymbol{\mathcal{M}}$ having a positive real part. For a given set of input-output coefficients J_{ij}, this imposes that firms productivities must be large enough, otherwise no realisable equilibrium exists.

- For all finite values of q in the CES production function, some analogous conditions must be fulfilled for a realisable equilibrium to exist [28].

- When $q = +\infty$ (i.e. in the Cobb-Douglas case), positive solutions to the equilibrium equations *always exist*, independently of productivities or network coefficients [27].

The possible non-existence of static solutions for generic production functions and network topologies compels us to go beyond equilibrium and formulate dynamical equations that would still make sense in such cases. But even in situations where such an equilibrium exists, it is by no means automatic that the economy is able to reach it on its own device. And even if it does, the description of non-adiabatic situations, i.e. those for which technologies and productivities evolve on a time shorter than the time needed to reach equilibrium, also require consistent dynamical equations. Interestingly, when the economy is close to an instability, e.g. when the smallest eigenvalue of $\boldsymbol{\mathcal{M}}$ tends to zero in the Leontief case, the time needed to reach equilibrium becomes infinitely large. This not only makes the adiabatic assumption moot but, as we shall see, compels the modeller to handle dynamical effects with special care.

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[5] Vector division in Eq. (II.6b) is understood as component-wise division.

III. A FIRST "NAIVE" APPROACH

In this section, we introduce the simplest version of a dynamical model aimed as describing out-of-equilibrium effects (transient or permanent) in a network economy. The equations we will postulate are based on reasonable "rules of thumb" that firm decision makers are likely to use in real life conditions [35–37]. There is obviously still a demarcation line between purists, who insist that these decision rules must be based on rational, forward looking optimisation programs, and a growing cohort of pragmatists who believe that modellers should embrace radical uncertainty and adopt behavioural rules closer to reality, with enough flexibility to avoid absurd paradoxes and accommodate, at least to some extent, Lucas' critique [38].

In looking for such dynamical equations, we draw inspiration from what physicists call "phenomenological approaches", heavily based on symmetry and genericity arguments. Such arguments allow one to avoid getting lost in the "wilderness" of possible models – to paraphrase Sims – once the straight-jacket of rationality is jettisoned. As we have learnt from physics, general arguments can often be used to write down correct equations before the underlying foundations have been worked out. For example, the Navier-Stokes equations for fluid motion have been postulated in the XIXth century based on general arguments, 50 years before Boltzmann's statistical theory of molecular motion gave a solid, first principle justification of these equations.

A. Forces Restoring Equilibrium

Whereas in economic equilibrium profits are zero and markets clear, out-of-equilibrium situations means, tautologically, non zero profits and/or excess supply or demand. So we naturally introduce, for each firm, two indicators that measure the distance from equilibrium: $\mathscr{E}_i(t)$ is the excess production at time t (interpreted as unsatisfied demand if $\mathscr{E}_i(t) < 0$), and $\mathscr{P}_i(t)$ the instantaneous profit or losses of the firm at time t. Prices and productions will then adapt, through some kind of *tâtonnement* to reduce these discrepancies. Faced with excess production, firms will lower prices to prop up demand, and/or reduce production to limit losses. Faced with excess demand, on the other hand, firms can consider increasing prices and/or increase production. Similarly, when profits are positive, firms can be tempted to increase production but at the same time competition, attracted by the prospect of a profit, should put pressure on prices. If profits are negative, firms will try to adapt by lowering production and increase prices, with the hope of better compensating production costs.

All these rules are common sense and it is hard to argue that they are not at play in the real economy. What is more debatable, however, is how to model them quantitatively. In this work, we further assume that all these effects are *linear* in $\mathscr{E}_i(t), \mathscr{P}_i(t)$, at least when these imbalances are small enough. It is also reasonable to think in terms of relative, non-dimensional quantities, i.e. ratios of $\mathscr{E}_i(t)$ to total production $z_i\gamma_i(t)$ and $\mathscr{P}_i(t)$ to total sales $z_i\gamma_i(t)p_i(t)$. Hence we write our *tâtonnement* rules as:

$$\log\left(\frac{p_i(t+\delta t)}{p_i(t)}\right) = \left(-\alpha\frac{\mathscr{E}_i(t)}{z_i\gamma_i(t)} - \alpha'\frac{\mathscr{P}_i(t)}{z_i p_i(t)\gamma_i(t)}\right)\delta t \tag{III.1a}$$

$$\log\left(\frac{\gamma_i(t+\delta t)}{\gamma_i(t)}\right) = \left(\beta\frac{\mathscr{P}_i(t)}{z_i p_i(t)\gamma_i(t)} - \beta'\frac{\mathscr{E}_i(t)}{z_i\gamma_i(t)}\right)\delta t, \tag{III.1b}$$

where δt is an elementary time step, and the parameters $\alpha, \alpha', \beta, \beta'$ characterize the speed of adjustment in the face of imbalances. From our general arguments above, we expect that all these parameters are non-negative, i.e. that firm policies and market forces tend to dampen imbalances. Whether these will be sufficient to stabilize the whole economy around the classical equilibrium described in the previous section is the whole point of the present research.

Note that we could have chosen these parameters to depend on the firm i, some firms choosing to be more aggressive than others in their adjustment policy. In the present work we will stick to firm independent values for $\alpha, \alpha', \beta, \beta'$. Finally, the simple rules Eqs. (III.1a, III.1b) are very similar in spirit to rules used in several well studied Agent Based Models – see [22, 26].

B. Dynamical Equations

Now, Eqs. (III.1a, III.1b) are closed by expressing imbalances in terms of prices p_i and productions π_i, as:

$$\mathscr{P}_i(t) = p_i(t)\pi_i(t) - \sum_{j=1}^{N} Q_{ij}(t)p_j(t) - p_0(t)\ell_i(t) = \gamma_i(t)\left(z_i p_i(t) - \sum_{j=1}^{N} J_{ij}p_j(t) - J_{i0}p_0(t)\right) \tag{III.2a}$$

$$\mathscr{E}_i(t) = \pi_i(t) - \sum_{j=1}^{N} Q_{ji}(t) - C_i(t) = z_i\gamma_i(t) - \sum_{j=1}^{N} J_{ji}\gamma_j(t) - C_i(t), \tag{III.2b}$$

where $C_i(t)$ is the consumption of households, $\ell_i(t)$ the quantity of labour, and where we again stick to constant return to scale Leontief production functions.

Finally, one must model the consumption of households. For simplicity, we assume here that households work full time, with L being the total amount of labour (this assumption will be relaxed below, as we will allow for unemployment). Consumption is obtained by saturating the current budget $p_0(t)L$ to maximize a log-consumption utility, i.e.

$$\max_{\mathbf{C}(t)} \boldsymbol{\theta} \cdot \log \mathbf{C}(t) \qquad \text{with} \qquad \mathbf{p}(t) \cdot \mathbf{C}(t) \leq p_0(t) \sum_i \ell_i(t), \tag{III.3}$$

where θ_i is the preference for good i. The optimal consumption is then $C_i(t) = \mu(t)\theta_i/p_i(t)$ with $\mu(t) = p_0(t)L_0/\sum_i \theta_i$.

Putting all these ingredients together and taking the continuous time limit $\delta t \to 0$ yields the following system of coupled non-linear ordinary differential equations:

$$z_i\gamma_i(t)\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\alpha p_i(t)\left(\sum_j \mathcal{M}_{ji}\gamma_j(t) - \frac{\mu(t)\theta_i}{p_i(t)}\right) - \alpha'\gamma_i(t)\left(\sum_j \mathcal{M}_{ij}p_j(t) - V_i\right) \tag{III.4a}$$

$$z_i p_i(t)\frac{\mathrm{d}\gamma_i}{\mathrm{d}t} = \beta\gamma_i(t)\left(\sum_j \mathcal{M}_{ij}p_j(t) - V_i\right) - \beta' p_i(t)\left(\sum_j \mathcal{M}_{ji}\gamma_j(t) - \frac{\mu(t)\theta_i}{p_i(t)}\right). \tag{III.4b}$$

C. Perturbations Around Equilibrium

Equations (III.4) are the "naive" candidate equations for the out-of-equilibrium dynamics of the firm network model. One immediately checks that injecting the equilibrium solutions $p_{\mathrm{eq},i}$ and $\gamma_{\mathrm{eq},i}$ (given by Eqs. (II.6a, II.6b)) cancels out the right hand sides of these equations, as it should be. One can also study the linear stability of this equilibrium. Writing $p_i(t) = p_{\mathrm{eq},i} + \delta p_i(t)$ and $\gamma_i(t) = \gamma_{\mathrm{eq},i} + \delta\gamma_i(t)$ and keeping only terms of order 1 in $\delta(.)$, one finds a linear evolution equation for a $2N$ dimensional vector $\mathbf{U} = (\delta\mathbf{p}, \delta\gamma)$, of the form:

$$\frac{\mathrm{d}\mathbf{U}(t)}{\mathrm{d}t} = \mathbb{D}\mathbf{U}(t). \tag{III.5}$$

The equilibrium stability is determined by the sign of the eigenvalues of the corresponding $2N \times 2N$ dynamical matrix $\mathbb{D}$. Such an analysis is provided in Appendix B.

When all eigenvalues are negative, equilibrium is locally stable. Any small perturbation away from equilibrium decays towards zero, at a rate asymptotically given by the eigenvalue closest to zero. The corresponding relaxation time τ_{relax} can be computed explicitly when the economy is on the verge of becoming un-realisable, i.e. when the smallest eigenvalue of the network matrix $\boldsymbol{\mathcal{M}}$ is at a distance $\varepsilon \to 0$ away from 0. We find:

$$\tau_{\mathrm{relax}} \approx \frac{2\max_j z_j}{\varepsilon} \times \begin{cases} \left(\alpha' + \beta' + \alpha - \sqrt{(\alpha'+\beta'+\alpha)^2 - 4(\alpha\beta + \alpha'\beta')}\right)^{-1} & \text{if} \quad (\alpha'+\beta'+\alpha)^2 > 4(\alpha\beta+\alpha'\beta') \\ (\alpha'+\beta'+\alpha)^{-1} & \text{if} \quad (\alpha'+\beta'+\alpha)^2 \leq 4(\alpha\beta+\alpha'\beta'). \end{cases} \tag{III.6}$$

This expression allows us to draw two important conclusions:

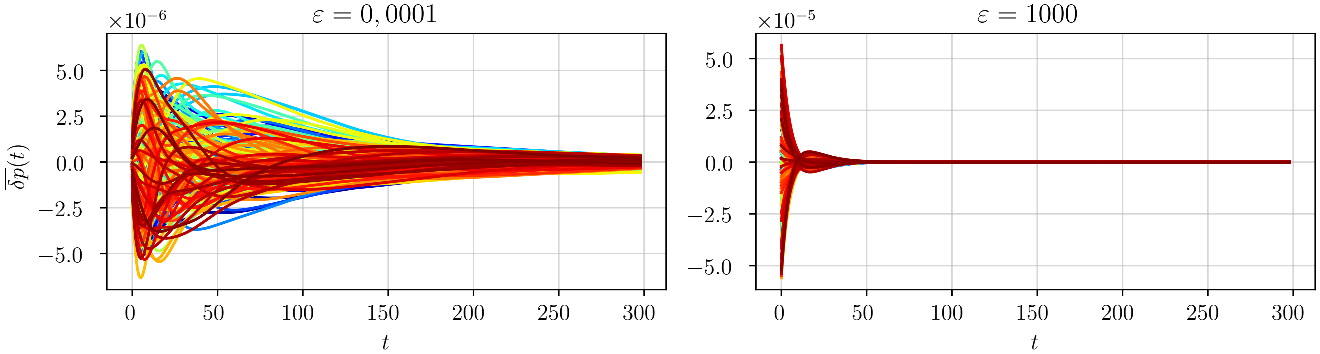

FIG. 1: Relative distance to equilibrium values of prices for the non-linear discrete dynamics Eqs. (III.4) for $N = 100$ firms. The initial relative distance in the simulation is taken to be $\delta = 10^{-3}$. The high productivity regime corresponds to a high value of $\varepsilon = 1000$ and leads to a very short relaxation time τ_{relax}. On the other hand, in the low productivity regime where $\varepsilon \to 0$, the system takes longer and longer to reach equilibrium again, and the relaxation time τ_{relax} diverges.

- When $\varepsilon \to 0$, the relaxation time of the system diverges, i.e. it takes an infinitely long time to reach equilibrium. As we mentioned in the introduction, this makes the adiabatic approximation unsuitable as changes in the technologies and in the network structure will happen before equilibrium can be reached. This long time scale also leads to an amplification of exogenous volatility in the system, see below.
- As long as α, α' or β' are strictly positive, the relaxation time is finite. The equilibrium is still stable if some coefficients are negative provided others are positive and sufficiently large.

A numerical illustration of the type of weakly out-of-equilibrium dynamics predicted by the model is shown in Fig. 1. One sees a complex interplay of spontaneous oscillations (coming from the imaginary part of the eigenvalues of the dynamical matrix $\mathbb{D}$) with a slowly decaying envelope, $\propto \exp(-t/\tau_{\text{relax}})$.

D. Excess Volatility

Note that if the parameters describing the economic equilibrium (such as productivities or household preferences, etc.) are slightly changing over time, the dynamical equation governing economic fluctuations, Eq. (III.5), becomes:

$$\frac{\mathrm{d}\mathbf{U}(t)}{\mathrm{d}t} = \mathbb{D}\mathbf{U}(t) + \xi(t), \tag{III.7}$$

where $\xi(t)$ represents the exogenous shocks to the economy. It is then not hard to show (see Appendix D) that in the limit $\varepsilon \to 0$, the volatility of prices and output is proportional to $\varepsilon^{-1/2}$, and can thus be much larger than the variance of the exogenous shocks when the system approaches the limit of stability.

The intuitive reason is that past shocks linger a very long time (comparable to τ_{relax}) in the system and aggregate with more recent shocks, leading to a much larger overall perturbation. Hence, the proximity to the point of instability is a natural candidate to explain the "small shocks, large business cycle" paradox (see [31] for a related discussion). An illustration of this phenomenon for our model is given in Fig. 2. However, such very long time persistence of fluctuations may be at odds with empirical data. We will discuss in section V C 3 below another scenario for "large business cycles" based on non-linear, *endogenous* fluctuations rather than on long-lived *exogenous* fluctuations.

E. Limitations

The above results suggest that, although "naive", our equations may be a reasonable starting point to describe the dynamics of firm networks, and actually provide an interesting generic scenario for anomalous fluctuations of output.

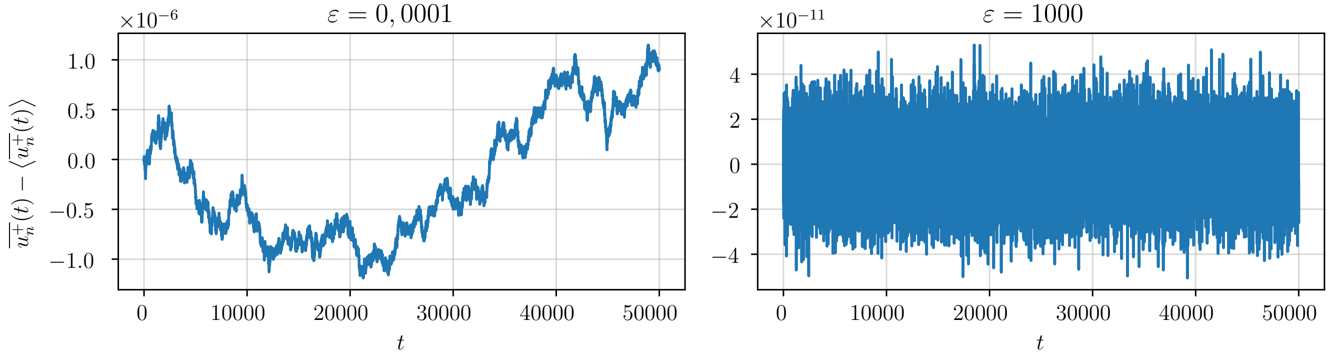

FIG. 2: Comparison of the components $u_N^+(t) = \langle U(t) \mid \sigma_N^+ \rangle$ yielding the volatility increase (as described in Appendix D) after productivity shocks with volatility $\sigma = 10^{-8}$ and $\varepsilon = 10^{-4}$, y-scale 10^{-6} (left), $\varepsilon = 10^3$, y-scale 10^{-11} (right). For $\varepsilon = 10^{-4}$, the volatility of output and prices is of the order of 10^{-6}, i.e. 100 times larger than σ, as expected from theory.

However, further numerical explorations of these equations reveal that the basin of attraction of the competitive equilibrium described in II is extremely narrow: only initial conditions less than 1% away from equilibrium lead to well behaved trajectories. Initial conditions that lie further away from equilibrium soon lead to a divergence of both prices and production levels, showing the limitations of the naive approach.

Another limitation concerns cases where the equilibrium is no longer defined, i.e. when $\varepsilon < 0$. In such a case, Eqs. (III.4) again cease to make sense (prices and productions are dragged below zero). Because of these impediments, our naive model cannot be usefully calibrated to empirical data, precisely because economic fluctuations are not small. As we will discuss in the next section, the only way to obtain well-behaved equations is to formulate the model such as to satisfy some incontrovertible constraints, i.e. causality (or "time to build") and physical bounds (consumption cannot be larger than production plus inventories).

IV. A FULLY CONSISTENT APPROACH

A. Imbalances and Causality

The naive approach of the previous section sweeps under the rug two important constraints, which are irrelevant at equilibrium: supply/demand imbalances (which are zero if markets clear) and causality (firms must decide production before they know how much they will manage to sell).

Accounting for the first implies the following. If demand exceeds supply, all of a firm's production will be sold and exchanged, whereas if supply exceeds demand, only the quantity that was demanded will be traded, leaving a surplus that will add to the firm's inventories. Hence, the flow of goods going from i to j must be computed with care; instead of the single quantity $Q_{ji}(t)$ considered in the previous section, we need to introduce the amount of goods i *demanded* by firm j, Q^{d}_{ji}, that can only be smaller or equal to the quantity actually exchanged, Q_{ji}. This can be understood as a contract that may only be fully honoured if firm i produces enough to meet all demands. In a similar fashion, we distinguish the amounts C^{d} demanded by households from what will be effectively sold to them, C, as well as the work hours posted by firms ℓ^{d}_i from the total amount of work L^{s} households are willing to provide. To handle the situation where supply exceeds demand, we keep track of firm i's inventory of good j, denoted by $I_{ij}(t)$ and to which we successively add the goods that the firms did not manage to sell or to use and subtract those that perished.

Implementing causality in the dynamics also means dissecting the firms' decision processes. Clearly, goods can only be sold at time t after they have been produced at time $t-1$, and prices may change (if only slightly) between these two times. More importantly, firms only have partial information about the amount of goods they will be able to buy and sell when they plan for the next production cycle. Likewise, the number of employees they will be able to hire is not known precisely, because it depends on the amount of work deemed acceptable by the households. It is at this stage that we will introduce a heuristic rule that allows firms to plan for the next production round by making

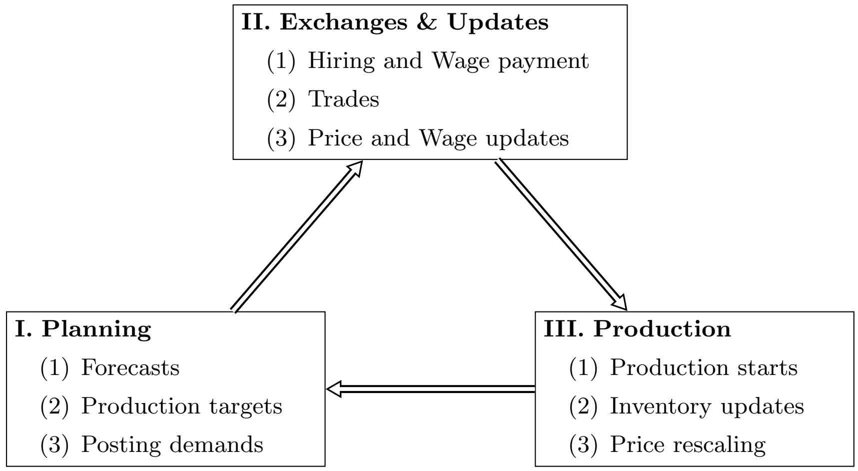

FIG. 3: Time-line of the model.

more or less informed guesses about these unknown quantities. In the present work, we assume that firms base their estimate on what happened in the previous time step, although more complicated and more general rules can already be imagined.

B. Time-line

In order to keep all causal constraints satisfied, one must carefully set up a consistent chronology for the actions of firms and households. The resulting time-line of the model is schematized in Fig. 3. Each time step δt ($\delta t = 1$ hereafter) is conveniently sliced in three successive "epochs", represented as boxes in Fig. 3. At the end of time step $t-1$, goods have been produced and are available for consumption at t in quantities $\pi_i(t)$ and prices $p_i(t)$.

1. Planning

At any given time, firms must plan how much to produce for the following period. To capture this, we keep the same *tâtonnement* rule as in the naive version of our model, Eq. (III.1b), but using now the *expected* profits $\mathbb{E}_t[\mathscr{P}_i]$ and excess productions $\mathbb{E}_t[\mathscr{E}_i]$ at the end of the period, which we specify below.

The target production available at $t+1$, $\widehat{\pi}_i(t+1)$, is set using

$$\log\left(\frac{\widehat{\pi}_i(t+1)}{\pi_i(t)}\right) = 2\beta \frac{\mathbb{E}_t[\mathscr{P}_i(t)]}{\mathbb{E}_t[\mathscr{G}_i(t)] + \mathbb{E}_t[\mathscr{L}_i(t)]} - 2\beta' \frac{\mathbb{E}_t[\mathscr{E}_i(t)]}{\mathbb{E}_t[\mathscr{S}_i(t)] + \mathbb{E}_t[\mathscr{D}_i(t)]}, \tag{IV.1}$$

where $\mathscr{G}_i(t)$ denotes the proceeds of the sales ("gains"), $\mathscr{L}_i(t)$ the production costs ("losses") and $\mathscr{D}_i(t)$ the overall demand for good i. We underline that since the available amount of good i is already known to the firm at time t, one has $\mathbb{E}_t[\mathscr{S}_i(t)] \equiv \mathscr{S}_i(t)$.

Once the target productions for $t+1$ are decided, the corresponding quantities $\widehat{Q}_{ij}$ are computed according to Eq (II.3). Firm i then posts its demands for inputs j for delivery at time t, taking into account their current stock of I_{ij} of said inputs, with the rule[6]

$$Q^{\mathrm{d}}_{ij} = \begin{cases} \max\left(0, \widehat{Q}_{ij} - I_{ij}\right) & i = 1,\dots,N;\ j = 1,\dots,N \\ \widehat{Q}_{i0} & i = 1,\dots,N;\ j = 0. \end{cases} \tag{IV.2}$$

[6] A more complicated expression would have to be written in the general CES case with $b \neq 1$.

Thus, if stocks are plentiful, the firm will prefer drawing from them instead of buying new inputs. In the meantime, households calculate their own consumption target for good i as detailed below and they also decide, given offered wages, how much labour they are willing to supply, a quantity we call $L^{\mathrm{s}}(t)$ that may now not correspond to full employment.

2. Exchanges & Price/Wage Updates

At this point, firms start hiring workers from the job market, albeit without exceeding the total supply of work L^{s}, i.e.

$$\ell_i(t) = \ell_i^{\mathrm{d}}(t) \min\left(1, \frac{L^{\mathrm{s}}(t)}{L^{\mathrm{d}}(t)}\right); \qquad L^{\mathrm{d}}(t) := \sum_i \ell_i^{\mathrm{d}}(t), \tag{IV.3}$$

where ℓ_i is the real amount of work contracted by firm i. Workers are paid the same wage $p_0(t)$ independently of their employer.[7] Conventionally, we prescribe that wages are paid immediately upon hiring – regardless of any technical unemployment in the future caused by shortages of inputs – which allows the household to compute its available budget for the present period:

$$B(t) = S(t) + p_0(t) \sum_i \ell_i(t). \tag{IV.4}$$

The household's demands for goods $C_i^{\mathrm{d}}(t)$ are computed in section IV D.

Trading can now start, whereby firms sell their production and buy the goods they need, in a way to satisfy the constraint that the total amount of goods sold cannot exceed production plus inventory, viz.

$$C_i(t) + \sum_j Q_{ji}(t) \leq \pi_i(t) + I_{ii}(t) := \mathscr{S}_i(t). \tag{IV.5}$$

If demand exceeds supply, buyers are satisfied proportionally to their posted demand, and so quantities Q that are effectively exchanged are given by

$$Q_{ji}(t) = Q_{ji}^{\mathrm{d}}(t) \min\left(1, \frac{\mathscr{S}_i(t)}{\mathscr{D}_i(t)}\right); \qquad \mathscr{D}_i(t) := C_i^{\mathrm{d}}(t) + \sum_j Q_{ji}^{\mathrm{d}}(t), \tag{IV.6}$$

where $\mathscr{D}_i(t)$ is the total demand for good i at time t. The equation for $C_i(t)$ is slightly more convoluted because we do not give households access to debt, see Eq. (IV.30) below.

At this point, firms have an exact knowledge of their earnings and expenses. Their profit at round t may now be computed:

$$\mathscr{P}_i(t) = p_i(t)\left(\sum_j Q_{ji}(t) + C_i(t)\right) - \left(\sum_j p_j(t) Q_{ij}(t) + p_0(t)\ell_i(t)\right) := \mathscr{G}_i(t) - \mathscr{L}_i(t), \tag{IV.7}$$

Firms also know how much excess supply or demand they actually registered:

$$\mathscr{E}_i(t) = \mathscr{S}_i(t) - \mathscr{D}_i(t). \tag{IV.8}$$

Realised profits and supply/demand imbalances then generate price updates. We describe them exactly as in Eq. (III.1a), which now reads:

$$\log\left(\frac{p_i(t+1)}{p_i(t)}\right) = -2\alpha \, \frac{\mathscr{E}_i(t)}{\mathscr{S}_i(t) + \mathscr{D}_i(t)} - 2\alpha' \, \frac{\mathscr{P}_i(t)}{\mathscr{G}_i(t) + \mathscr{L}_i(t)}, \tag{IV.9}$$

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[7] Extending the model to firm-dependent wages would be interesting but requires one to move beyond a representative agent description of the household sector.

where now all quantities are known.[8]

Prices are updated because of tension between supply and demand. By the same token, tensions on the job market are bound to lead to wage updates, which we postulate to be of the same form as for price updates, namely

$$\log\left(\frac{p_0(t+1)}{p_0(t)}\right) = 2\omega\,\frac{L^{\rm d}(t)-L^{\rm s}(t)}{L^{\rm d}(t)+L^{\rm s}(t)}, \qquad \text{(IV.10)}$$

meaning that excess demand of labour increases wages, and vice-versa. This rule implements a Phillips curve at each time step [39, 40]. One could also use an asymmetric update rule, accounting for the fact that lowering nominal wages is more difficult than raising them. Finally, one could also consider adding a direct coupling between the inflation of the price of goods and wages, as an extra term in the right hand side of Eq. (IV.10).

3. Production

The last epoch corresponds to the start of production. Firm i uses the workforce ℓ_i, along with available quantities $Q^{\rm a}_{ij}$ that depend on exchanges Q, optimal inputs $\widehat{Q}$ and inventories I, as

$$Q^{\rm a}_{ij}(t) = Q_{ij}(t) + \min\left(I_{ij}, \widehat{Q}_{ij}\right). \qquad \text{(IV.11)}$$

Indeed, if the inventory I allows to provide for optimal input $\widehat{Q}$, then no demand is posted (see Eq. (IV.2)): $Q = 0$ and $Q^{\rm a} = \widehat{Q}$. Otherwise, the firm acquired a quantity Q that now adds to available stocks, and so $Q^{\rm a} = Q + I \leq \widehat{Q}$. Note that labour cannot be stored, and therefore $I_{i0} = 0$ at all times.

Now that all of the available inputs $Q^{\rm a}_{ij}$ and labour ℓ_i are known, the outputs are determined by the firms' production functions, which in the Leontief case with $b = 1$ entails:

$$\pi_i(t+1) = z_i(t)\min\left[\min_j\left(\frac{Q^{\rm a}_{ij}(t)}{J_{ij}}\right), \frac{\ell_i(t)}{J_{i0}}\right]. \qquad \text{(IV.12)}$$

The firms' inventories of their own production is also updated, as

$$I_{ii}(t+1) = e^{-\sigma_i}\left(\pi_i(t) + I_{ii}(t) - \sum_j Q_{ji}(t)\right), \qquad \text{(IV.13)}$$

where the decay factor σ_i measures the perishability of good i. For durable goods, $\sigma_i \ll 1$ and $e^{-\sigma_i} \approx 1$, whereas $\sigma_i \gg 1$ and $e^{-\sigma_i} \ll 1$ for perishable goods.

Furthermore, in the Leontief framework total production is limited by the scarcest input, which is therefore depleted during production, leaving a fraction of the other inputs unused. We denote

$$j^\star = \arg\min_j\left(\frac{Q^{\rm a}_{ij}}{J_{ij}}\right),$$

so that we can write the fraction of inputs $k \neq j^\star$ effectively used as

$$Q^{\rm u}_{ik}(t) = \frac{J_{ik}}{J_{ij^\star}}Q^{\rm a}_{ij^\star}. \qquad \text{(IV.14)}$$

The remainder of these unspent inputs goes to firm inventories, and their update may be written using Eq. (IV.11), as

$$I_{ik}(t+1) = e^{-\sigma_k}\left(Q^{\rm a}_{ik} - Q^{\rm u}_{ik}\right). \qquad \text{(IV.15)}$$

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[8] Since markets do not clear and profits are non zero, we choose symmetric normalisation factors involving the average of supply and demand for the first term, and the average of sales and costs for the second.

Finally, for numerical purposes, it is convenient to rescale new prices $p_i(t+1)$ by the new wage $p_0(t+1)$ to avoid exponential growth (or decay) of prices induced by inflation (or deflation), effectively measuring prices in units of wages. We therefore set:

$$p_i(t+1) \longrightarrow \frac{p_i(t+1)}{p_0(t+1)}; \qquad p_0(t+1) \longrightarrow 1. \tag{IV.16}$$

[Note that profits and savings should also be appropriately rescaled, when necessary, e.g. $S(t+1) \to S(t+1)/p_0(t+1)$, etc.]

This concludes the third and last epoch of the time step. The process is then repeated at time $t+1$, with productions $\pi_i(t+1)$ and prices $p_i(t+1)$.

To close the model, we now need to specify how firms estimate their future profits/losses and excess/deficit production. The behaviour of households must also be spelled out, to allow for the determination of the demand of goods and the supply of labour.

C. Expected Profits and Imbalances

We may write the expected profit of firm i as

$$\mathbb{E}_t[\mathscr{P}_i] = p_i(t)\left(\sum_j \mathbb{E}_t[Q_{ji}] + \mathbb{E}_t[C_i]\right) - \left(\sum_j p_j(t)\mathbb{E}_t[Q_{ij}] + p_0(t)\mathbb{E}_t[\ell_i]\right), \tag{IV.17}$$

showing that in the planning phase firms must estimate future goods and labour demand, which we will denote generically as $\mathbb{E}_t[Q]$. Similarly, the expected excess production is also a function of $\mathbb{E}_t[Q]$:

$$\mathbb{E}_t[\mathscr{E}_i] = \pi_i(t) + I_{ii}(t) - \sum_j \mathbb{E}_t[Q_{ji}] - \mathbb{E}_t[C_i]. \tag{IV.18}$$

The simplest assumption we can adopt is that firms are sticky, and estimate all future demands to be equal to their last observation (which follows the rationale that they produced in order to meet total demand), i.e.

$$\mathbb{E}_t[Q] = Q^{\mathrm{d}}(t-1). \tag{IV.19}$$

This is the rule that we will explore in the present paper, but some immediate generalisations come to mind: one is that firms may factor in *realized* quantities $Q(t-1)$ in their estimate, and set

$$\mathbb{E}_t[Q] = \lambda Q^{\mathrm{d}}(t-1) + (1-\lambda)Q(t-1), \tag{IV.20}$$

where $\lambda \in [0,1]$ is a parameter, set to $\lambda = 1$ henceforth in our "sticky" assumption.

The second possible generalisation is that firms may use a more sophisticated learning rule that allows them to estimate $\mathbb{E}_t[Q]$ using time-series analysis, the simplest of which is "constant gain learning" (equivalent to computing the exponential moving average) of past realized demands. Trend-following, extrapolative rules may also be considered. These extensions are beyond the scope of the present paper; at this stage, our ambition is to set up a minimal consistent framework, free of numerical instabilities, that can be calibrated to micro-data.

D. Household Demand and Labour

1. *Work-elastic Households*

As in standard macroeconomic models, we assume that households are represented by a single representative agent with a certain disutility for work, who seeks to maximize the following utility function[9]

$$\mathcal{U}(t) = \sum_j \theta_j \log C_j(t) - \frac{\Gamma}{1+\varphi}\left(\frac{L(t)}{L_0}\right)^{1+\varphi}, \tag{IV.21}$$

[9] We restrict to a "myopic" optimisation here, that does not take into account the long-term forecasts and desires of the household. Inter-temporal effects would require to add interest rates, which we completely disregard in the present study.

where $L(t) = \sum_j \ell_j(t) := \sum_j Q_{j0}(t)$ is the total amount of work provided by the representative household. The so-called Frisch elasticity index φ [41] gives a measure of the convexity of the disutility of work, L_0 is the scale of the amount of work that the household is able to provide and Γ is a parameter that can be set to unity without loss of generality. In the limit $\varphi \to \infty$, firms are indifferent to the amount of work provided $L(t) < L_0$, but refuse to work more than L_0. With an utility function of this form, the household may then compute its optimal demand for good i, $C_i^{\mathrm{d}}(t)$ which it will set as a consumption target for period t, and the optimal amount of labour $L^{\mathrm{s}}(t)$ it is willing to provide to firms.

2. *The optimization sequence*

To compute the aforementioned quantities, the household needs to know its current savings $S(t)$ and anticipate its income for the next period. The expected utility is estimated with optimistic forecasts (i.e. consumption demand will be met and available labour will be fully utilized). Wage $p_0(t)$ and prices $p_i(t)$, on the other hand, are all known before the "Exchange and Update" stage, see IV B 2. Hence,

$$\mathbb{E}_t[\mathcal{U}] = \sum_i \theta_i \log C_i^{\mathrm{d}}(t) - \frac{1}{1+\varphi}\left(\frac{L^{\mathrm{s}}(t)}{L_0}\right)^{1+\varphi}, \tag{IV.22}$$

with an expected budget constraint that reads[10]

$$\sum_i p_i(t) C_i^{\mathrm{d}}(t) = p_0(t) L^{\mathrm{s}}(t) + S(t) := \mathbb{E}_t[B], \tag{IV.23}$$

where $\mathbb{E}_t[B]$ is the expected (or in fact hoped for!) budget. For convenience, we denote as $W_0 = p_0 L_0$ the wage associated to L_0 work-hours.

The household optimizes its expected utility while enforcing the budget constraint using a Lagrange multiplier $\mu(t)/W_0$, so that

$$C_i^{\mathrm{d}}(t) = L_0 \frac{\theta_i}{\mu(t)} \frac{p_0(t)}{p_i(t)} \tag{IV.24a}$$

$$L^{\mathrm{s}}(t) = L_0\, \mu(t)^{1/\varphi}. \tag{IV.24b}$$

In order to find $\mu(t)$, one must enforce (IV.23). We find the following equation on $\mu(t)$:

$$\mu^k(t) + \frac{S(t)}{W_0(t)}\, \mu(t) = \bar{\theta}, \tag{IV.25}$$

with $k = 1 + 1/\varphi$ and $\bar{\theta} = \sum_i \theta_i$. For instance, if $\varphi = \infty$ (constant work offer $L^{\mathrm{s}}(t) = L_0$), we have

$$\mu(t) = \frac{\bar{\theta} W_0(t)}{W_0(t) + S(t)}. \tag{IV.26}$$

When $\varphi = 1$ (a common value found in the literature and corresponding to a quadratic work-disutility), we have

$$\mu(t) = \frac{1}{2W_0(t)}\left(\sqrt{S(t)^2 + 4\bar{\theta} W_0(t)^2} - S(t)\right). \tag{IV.27}$$

We highlight that, because of possible unemployment, the household may want to consume more than it is able to spend when $L^{\mathrm{d}}(t) < L^{\mathrm{s}}(t)$.

A final word on the scaling behaviour of these quantities with N is in order. For large N we expect that the size of the household will also be of order N. Noting that $\bar{\theta}$ is of the same order, one finds the following consistent scalings if we assume that $L_0 \sim \sqrt{N}$:

$$\mu \sim \sqrt{N}; \qquad L^{\mathrm{s}} \sim N; \qquad C_i^{\mathrm{d}}(t) \sim 1, \tag{IV.28}$$

meaning that total work-hours and total consumption are proportional to the size of the population, as it should be.

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[10] In a follow-up paper [42] we shall introduce precautionary savings and interest rates, which lead to the appearance of inflationary equilibria.

3. *Confidence Effects*

In the setup above, households consume regardless of the state of the economy. Although not necessary for the purpose of the present paper, we believe it is important to introduce a notion of confidence in the economy by coupling the consumption propensity to the unemployment level, taken as a proxy of consumer confidence. Hence we allow the utility of consumption to vary as:

$$\log\left(\frac{\theta_i(t)}{\theta_i^0}\right) = 2\omega' \, \frac{L^{\mathrm{d}}(t) - L^{\mathrm{s}}(t)}{L^{\mathrm{d}}(t) + L^{\mathrm{s}}(t)}, \tag{IV.29}$$

where θ_i^0 are the baseline values for consumption preferences.

In a booming economy where demand for workforce is high, households will tend to consume more (increased θ's); whereas it will consume less in a failing economy with high unemployment.

4. *Savings Update*

Because we do not allow households to borrow, real consumption must be adjusted in the case of partial unemployment. In this case, the available budget is necessarily smaller that what was hoped, leading to a realized consumption:[11]

$$C_i^{\mathrm{r}}(t) = C_i(t) \min\left(1, \frac{B(t)}{\sum_j p_j(t) C_j(t)}\right); \qquad C_i(t) = C_i^{\mathrm{d}}(t) \min\left(1, \frac{\mathscr{S}_i(t)}{\mathscr{D}_i(t)}\right), \tag{IV.30}$$

with $B(t)$ their available budget computed in (IV.4) . The difference between $C_i(t)$ and $C_i^{\mathrm{r}}(t)$, if positive, is added to the inventory $I_{ii}(t)$ of firm i. The households' savings are then updated as:

$$S(t+1) = B(t) - \sum_i p_i(t) C_i^{\mathrm{r}}(t). \tag{IV.31}$$

E. Discussion

The above steps look rather tedious and considerably more complex than the simple logic behind our first "naive" model. Nonetheless, they are quite natural when one decomposes all the stages of a real production process. But more importantly, we have found that short-circuiting any of these steps leads to inconsistent dynamics with spurious instabilities, reflecting that natural constraints are in fact violated.

An important difference with the naive version of section III is the large number of update rules that involve non-linearities, such as those involving taking the maximum or minimum of two expressions. Furthermore, the number of thumb rules used by firms and households to aid their decision has increased, and so has the number of parameters that describe a given instance of our toy economy.

Therefore, and in spite of the fact that the naive model permits a reasonably good understanding of certain regions of the parameter-space of the full model, we cannot reasonably attempt an exhaustive description using analytical tools only. We must therefore resort to a numerical exploration of its properties, using computer simulations that are described in detail in the pseudo-code provided in Appendix E. We also provide access to an open access simulation tool that allows the reader to explore different configurations here: `https://yakari.polytechnique.fr/dash`.

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[11] An extension could be imagined, where workers borrow money to compensate the gap between the expected and realised budget, but we do not consider this in the present version of our model.

V. AN EXPLORATORY NUMERICAL STUDY

The following section is an early attempt to describe some of the very rich phenomenology this model can produce. Because of the many possible parameter configurations we only scratch the surface here, and leave a more detailed account for a later communication.

To facilitate reading this section, we will first recall the different parameters we can adjust. We will then explore the different types of dynamical trajectories that can be observed in our toy economy, and classify them into different "phases". This idea comes from physics, where the macroscopic properties of a system can be split into different regions where its aggregate behaviour is qualitatively the same. These regions only depend on the values taken by a handful of parameters that describe the system; an eloquent example is that of water, which depending on the pressure or temperature can be in either the liquid, solid or gas phase.

We will therefore present in the following "phase diagrams" that summarize the influence of the parameters on the broad dynamical behaviour of our model, an idea that was already advocated for economic Agent-Based Modelling in [26].

A. Summary of Parameters

The different parameters introduced in the previous sections may be split into two categories: static parameters, describing the production network and the production function, and dynamic parameters, describing the evolution of prices, labour and outputs. We provide an overview of them and of the typical values we assign to them in our simulations below.

a. Static Parameters

1. Number of firms N – here $N = 100$.
2. Type of network – here a random regular directed network [43, 44], where each firm has the same number of clients and suppliers $d = 15$.
3. CES production function – here a Leontief production function ($q = 0^+$) with a return to scale parameter $b = 0.95$.[12]
4. The smallest eigenvalue ε of the production matrix $\boldsymbol{\mathcal{M}}$, which for large values corresponds to a presumably stable economy.
5. Firm inter-linkages J_{ij}, which we all take to be 1 when firms i and j are linked and zero otherwise.
6. Firm productivities z_i, first set to 1 and then adapted to adjust ε to take the required value.[13]
7. Household consumption preferences θ_i^0, modelled by iid uniform random variables rescaled to have $\sum_i \theta_i^0 = 1$.
8. Work disutility Frisch index, set to $\varphi = 1$ (quadratic disutility of labour) and scale of workforce set to $L_0 = 1$.
9. The behavioural parameter λ, defined in Eq. (IV.20), is set to 1.

b. Dynamic Parameters

1. Parameters describing restoring forces: $\alpha, \alpha', \beta, \beta'$, (see Eqs. (IV.1)-(IV.9)). We restrict ourselves to the case $\beta' = \alpha' = \beta = \alpha$ and scan for varying values of α.
2. Phillips curve parameter, relating wages to tensions in the job market: ω (see Eq. (IV.10)).
3. Confidence parameter, relating consumption propensities to unemployment: ω' (see Eq. (IV.29)). For this study, we take $\omega' = \omega$.

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[12] Choosing b slightly below unity helps stabilising the dynamics and also prevents the relaxation time from diverging as the smallest eigenvalue of the production matrix $\varepsilon \to 0$. An in-depth discussion of this point will be provided in a follow-up paper.

[13] Modifying the productivity factors as $z' = z + \varepsilon - \min \mathrm{Sp}(\mathcal{M})$ makes the minimum eigenvalue of $\mathcal{M}$ equal to ε.

4. Perishability parameters σ_i describing the speed of decay of good i, all taken as $\sigma_i = \sigma$ except when otherwise indicated.

These choices therefore reduce the number of parameters to explore to four: $\varepsilon, \alpha, \omega$ and σ. We will now show how they may lead to a very rich phenomenology.

B. Perturbations Around Equilibrium

As stressed above, the naive model section III C could be linearised, and lead to a complete analytical estimation of the time needed to reach equilibrium. The non-linearities in the full-model, however, imply that perturbative analysis leads at best to piecewise-linear equations.[14]

To be precise, it is possible to attempt to linearise the different update rules by writing $\delta x(t) = x_{\mathrm{eq}} - x(t)$ for the perturbed value of any quantity x and developing the different equations to lowest order in $\delta\cdot$. For example, this procedure applied to the flows Q_{ji} reads:

$$\delta Q_{ji}(t) = \delta Q^{\mathrm{d}}_{ji}(t) + \min\left(0, \frac{\delta \mathscr{S}_i(t) - \delta \mathscr{D}_i(t)}{z_i \gamma_{\mathrm{eq},i}}\right). \tag{V.1}$$

Depending on the value of $\delta\mathscr{S}_i(t) - \delta\mathscr{D}_i(t)$, the system can be characterized by two different linear stability matrices. As the system evolves in time, and even for very small fluctuations around equilibrium values, it may switch back and forth between these two stability matrices, explaining some of the behaviour we observe.

We also stress that the model is well-defined even if we choose initial conditions very far from equilibrium. Preliminary studies show that initial conditions well above equilibrium values (up to 6 orders of magnitude) may still lead to an overall stable system. However, if the initial conditions are instead too small this can lead to divergences.

C. Phase Diagrams and Dynamical Types

For each set of values of the parameters (α, ω, σ, ε), we start from a random perturbation about equilibrium of relative magnitude $\delta = 10^{-3}$, taking e.g. $p_i(t) = p_{\mathrm{eq},i}(1 + \delta u)$ with u uniform in $[-1, 1]$.[15] We then run the dynamics for $T = 5000$ time-steps and consider only the last 2500 to classify the trajectory into one of several classes that are detailed below.

In general, we observe five types of behaviour or "phases": convergence towards the competitive equilibrium, convergence towards deflationary equilibria, crises, business-cycle like oscillations or chaotic oscillations and divergence, where the economy crashes after a finite number of time steps. Different phase diagrams corresponding to this classification can be seen on Figure 4 with ε in $[100, 10, 1, -5]$, and the study and description of these phases is detailed in the sections below.

We note in particular that:

- The region where the competitive equilibrium is reached shrinks as the economy approaches the instability $\varepsilon \to 0$ from above. When $\varepsilon < 0$, there is no equilibrium and only deflationary equilibria can be attained.
- For a fixed perishability σ one observes the following succession of phases as the restoring parameter α is increased: divergence when α is too small, followed by deflationary equilibria, then the reaching of the competitive equilibrium and finally cycles and chaos, corresponding to firms that are overly sensitive to imbalances.
- At the boundary between these phases, one can observe specific trajectories that we call crises, similar to the "tipping points" and "dark corners" described in [26, 46] – see below.

[14] This, as the general time-line framework outlined in section IV B, is a feature common to other ABMs, such as Mark-0 [26], or the one recently developed in [45].

[15] When $\varepsilon < 0$ and no competitive equilibrium can be defined, we start from random initial conditions between 1 and 2 for prices and productions.

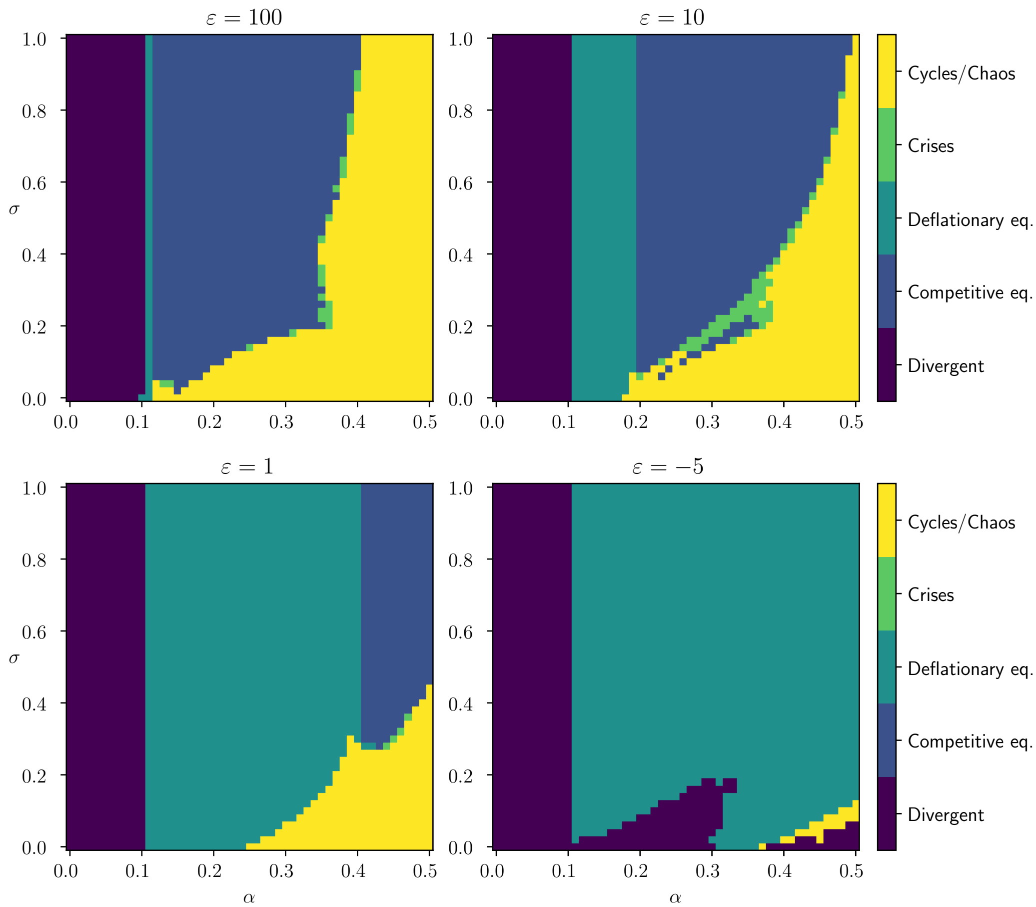

FIG. 4: Phase diagrams in the plane (α, σ) for the same network economy, $\omega = 0.1$ and different values of ε. The color code is explained in the legends. The region where the competitive equilibrium state is stable shrinks when ε decreases, and disappears when $\varepsilon < 0$ and deflationary equilibria take over. One also observes regions with cycles and chaos, and crises. Finally, when restoring forces are to weak (small α) the economy crashes.

The trajectory we will use to classify the behaviour of our model is that of relative price differences $\overline{\delta p}(t) := p(t)/p_{\mathrm{eq},i} - 1$.[16] In order to provide more vivid illustrations of some of these dynamical types, we have made firms slightly heterogeneous in their values of the parameters α and σ. In the figure captions below, the notation $\alpha, \sigma \in [A, B]$ means that these quantities are chosen uniformly in $[A, B]$, independently for each firm.

1. *Relaxation towards competitive equilibrium*

The most natural behaviour one would expect is for the economy to converge to a competitive equilibrium, where all profits are zero and markets clear, as classically assumed in economics models.

Within the corresponding phase, convergence can either be purely exponential, or correspond to damped oscillations or even damped chaos, see Fig. 5. The precise nature of the relaxation seems to depend on the relative time-scales for prices and production updates.

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[16] The trajectories of produced quantities are qualitatively similar within each phase, except that, as expected, high prices correspond to production troughs, and vice versa.

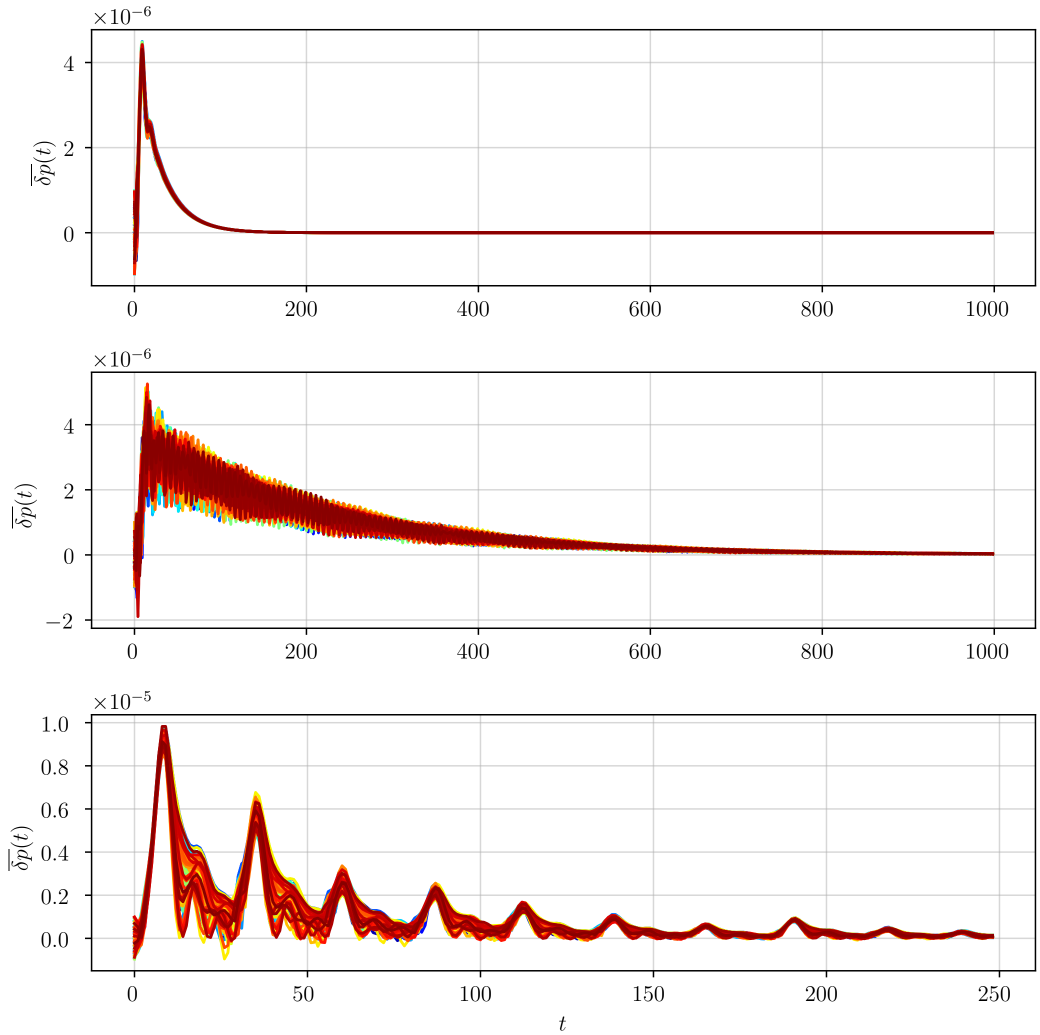

FIG. 5: Relaxation towards the competitive equilibrium after a perturbation of magnitude $\delta = 10^{-3}$. Top: Exponential relaxation for $\varepsilon = 10$, $\omega = \omega' = 0.2$, $\alpha = \alpha' = \beta = \beta' \in [0.6, 0.7]$ and $\sigma \in [0.5, 0.6]$. Middle: Damped oscillations for $\varepsilon = 1$, $\omega = \omega' = 0.2$, $\alpha = \alpha' = \beta = \beta' \in [0.8, 0.9]$ and $\sigma \in [0.2, 0.6]$. Bottom: Damped chaotic oscillations for $\varepsilon = 100$, $\omega = \omega' = 0.2$, $\alpha = \alpha' = \beta = \beta' \in [0.5, 0.6]$ and $\sigma \in [0.2, 0.6]$

2. *Relaxation towards deflationary equilibrium*

A very interesting feature of our model is the appearance of a different kind of equilibria, namely stationary points where profits and excess demand are non-zero, but equal to a constant value. We call them "deflationary" equilibria because prices synchronize with the inflation rate determined by the evolution of wages. This phase does not manifest itself when there is no wage-induced inflation ($\omega = 0$).

We denote by $\bar{\mathscr{P}}_i^\infty$ and $\bar{\mathscr{E}}_i^\infty$ the rescaled values of profits and excess supply/demand in the stationary state. These must then verify (see Eqs. (IV.1, IV.9)):

$$\begin{aligned} \alpha\bar{\mathscr{E}}_i^\infty + \alpha'\bar{\mathscr{P}}_i^\infty &= \omega\frac{L^{s,\infty} - L^{d,\infty}}{L^{s,\infty} + L^{d,\infty}} \\ \beta'\mathbb{E}_\infty[\bar{\mathscr{E}}_i] - \beta\mathbb{E}_\infty[\bar{\mathscr{P}}_i] &= 0. \end{aligned} \tag{V.2}$$

In this kind of equilibrium, forecasts of profits and imbalances are typically different from their realized counterpart. This discrepancy is mostly caused by an error in the forecast of realized consumptions introduced by a positive

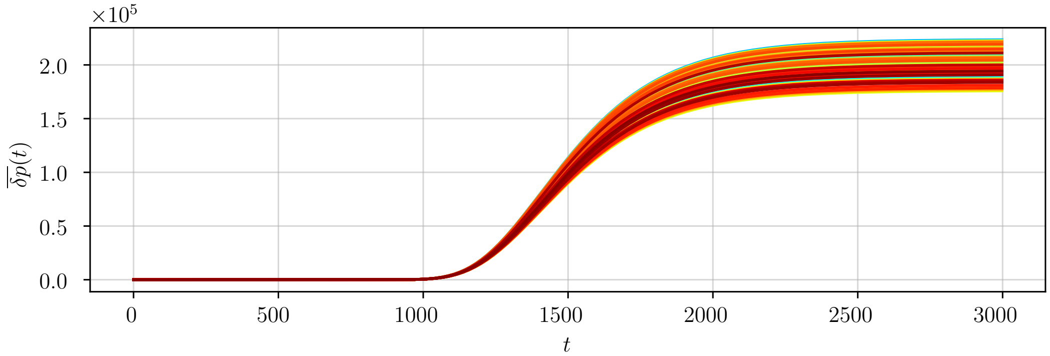

FIG. 6: Example of a deflationary equilibrium with $\varepsilon = 1$ and heterogeneous productivity factors. Note that we show here real prices (deflated by wages). We chose $\omega = \omega' = 0.2$, $\alpha = \alpha' = \beta = \beta' \in [0.5, 0.6]$ and $\sigma = 0.6$.

stationary imbalance in the labour market.

In contrast with the competitive equilibrium, which is independent of the dynamical parameter $\alpha, \alpha', \beta, \beta'$, deflationary equilibria are characterized by prices and production levels that depend on the parametrisation of the dynamics. Explicit expressions for the stationary prices/productions are, however, difficult to compute analytically.

Figure 6 shows an example of the convergence of inflation-adjusted prices towards their stationary values. In this preliminary study we have found equilibria to be rather stable.

These deflationary equilibria disappear when $\omega = 0$. The corresponding phase diagrams in that case are similar, when $\varepsilon > 0$, to those of Fig. 4, but with a wider region corresponding to the competitive equilibrium phase. When when $\varepsilon < 0$, the only possibilities are cycles/chaos (yellow phase) or a complete crash (black phase).

We underline finally that we have not found, within the present specification of the model, inflationary equilibria where the demand for labour exceeds the supply (although one can easily produce some non-Phillips inflation in competitive equilibria). However, introducing precautionary savings that yield non-zero interest rate leads to new phenomena, including a whole region where inflationary equilibria are found.

3. Oscillatory patterns

Owing to the strongly non-linear dynamics defining the model, it is natural to expect that some choices of the parameters lead – as in generic dynamical systems – to oscillations or to chaotic dynamics, which is indeed what we observe in a whole region of parameter space.

The first interesting oscillatory behaviour is that of spontaneously emerging business cycles, as shown in Fig. 7. They can be either synchronized (Fig. 7-a) or completely unsynchronized (Fig. 7-b), depending on the values of ω and ε, and the relative values of α and β'. Chaotic oscillations also emerge (see Fig. 7-c) [42].

We stress that such persistent oscillations, observed in the rather large portions of the phase diagram, are not due to external perturbations, absent in these simulations (compare with section III D where small external shocks are amplified by the proximity of an instability). Rather, this is a region of the phase diagram where the volatility of the economy is purely endogenous (see [31] for similar observations). This indicates yet another scenario to explain the "small shock, large business cycle" puzzle [3]: either because of the proximity of an unstable point, as in section III D, or because of the existence of self-sustained oscillations/chaos, as reported here and in many previous work in which a dynamical systems approach to economics was advocated, see e.g. [10, 16, 17, 47, 48] and also [13, 21, 26, 49] in the context of ABMs.

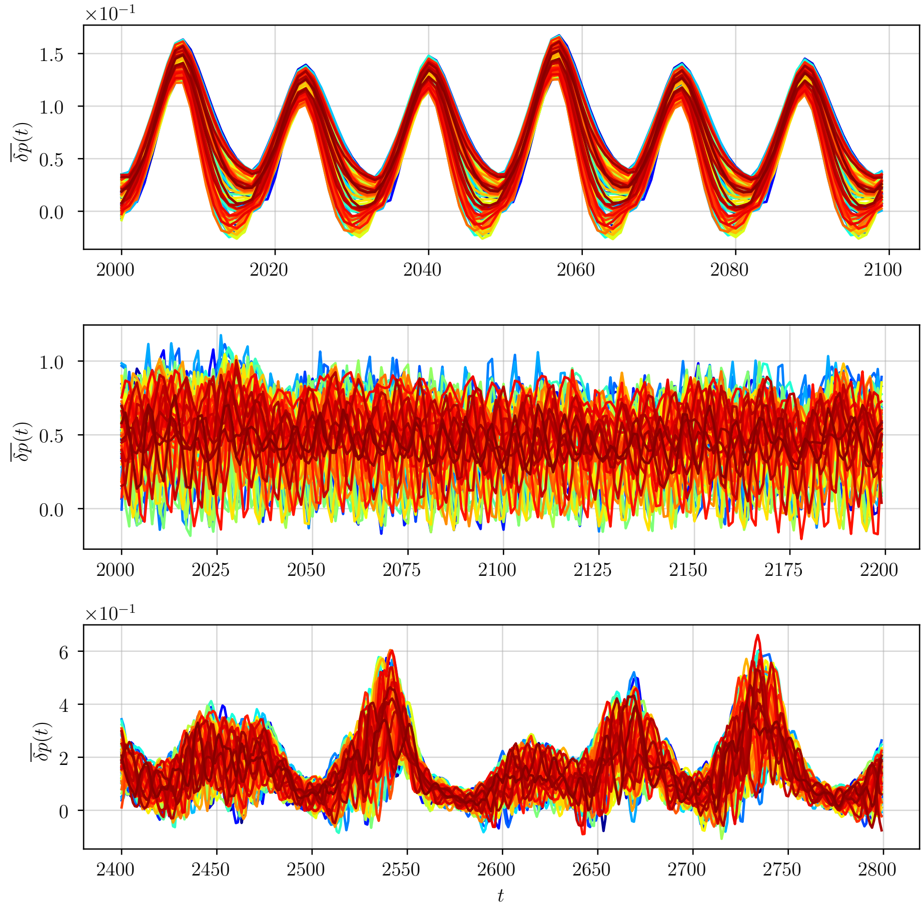

FIG. 7: Different types of price (or production) oscillations after an initial perturbation of magnitude $\delta = 10^{-3}$ from equilibrium. Top: Synchronized business cycles for $\varepsilon = 100$, $\omega = \omega' = 0.1$, $\alpha = \alpha' = \beta = \beta' \in [0.4, 0.5]$, $\sigma \in [0.1, 0.4]$. Middle: Unsynchronized oscillations for $\varepsilon = 100$, $\omega = \omega' = 0.2$, $\alpha = \alpha' = \beta \in [0.5, 0.8]$, $\sigma_{=}0.2$; $\beta' = 1.3\alpha$. Bottom: Chaotic oscillations for the same parameters except $\varepsilon = 1$ and $\beta' = 0.2\alpha$.

4. Intermittent Crises

This additional dynamical phase is represented in Figure 8. Here, a fast relaxation to equilibrium is followed by spontaneous destabilisation. The system enters a cycle of price inflation and plummeting production. This is most likely due to a switch between the stability matrices of the system, as discussed above in Eq. (V.1). The eigenvalues of the first matrix all have negative real parts, whereas the second has at least one eigenvalue with a positive real part, and therefore an unstable direction.

After this, non-linear saturation effects take over to quell the dynamics, and the system flows back towards equilibrium before the next crisis appears. These acute endogenous crises are one of the most interesting aspects of our model; they also appear in other Agent Based Models, see [26, 50] where they result from a generic synchronisation mechanism, see [51].

D. The unstable phase $\varepsilon < 0$

A weakness of the naive model of section III was that it consistently produced divergent trajectories whenever $\varepsilon < 0$. Our new model produces instead a wide range of behaviours in this case, from deflationary equilibria to rapid oscillations as illustrated on Figure 9. Of course, since there is no well-defined equilibrium, the convergent phase is

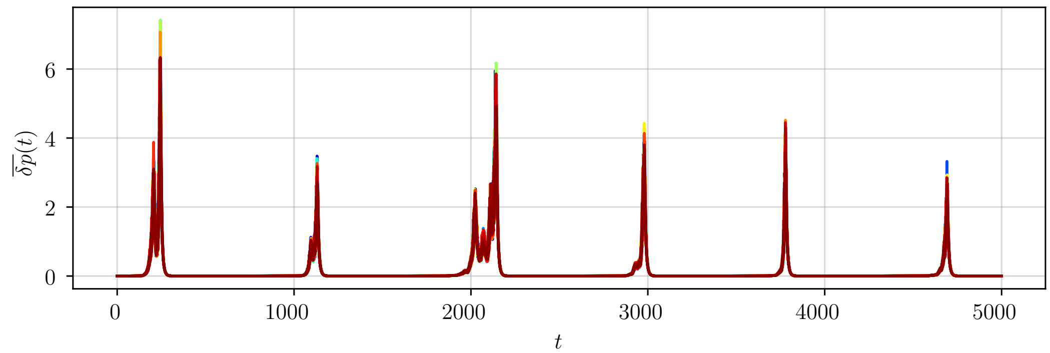

FIG. 8: Crises-like price pattern for $\varepsilon = 100$, $\omega = \omega' = 0.1$, $\alpha = \alpha' = \beta = \beta' = 1$, $\sigma = \infty$.

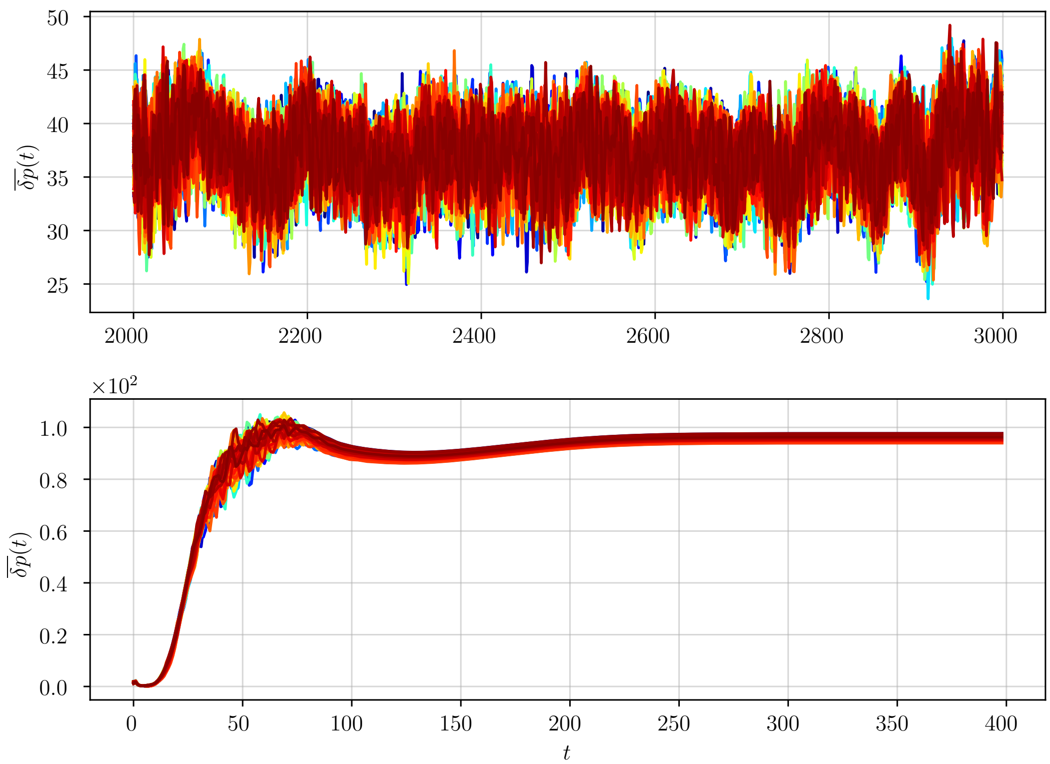

FIG. 9: Different possible price (or production) dynamics in the unstable phase $\varepsilon = -5$, for initial conditions for prices and productions randomly chosen between 1 and 2 times the equilibrium values. Top: Rapid oscillations for $\omega = \omega' = 0.02$, $\alpha = \alpha' = \beta = \beta' = 0.9$, $\sigma = 0.2$. Bottom: deflationary equilibrium for $\omega = \omega' = 0.02$, $\alpha = \alpha' = \beta \in [0.8, 0.9]$, $\sigma \in [0.2, 0.8]$.

now proscribed.

E. The Role of Perishability

Finally, we would like to illustrate the crucial role of inventories in determining the type of dynamics we observe. As shown in the phase diagrams of Figure 10 in the (α, ω) plane at fixed σ, goods that perish immediately ($\sigma = \infty$)

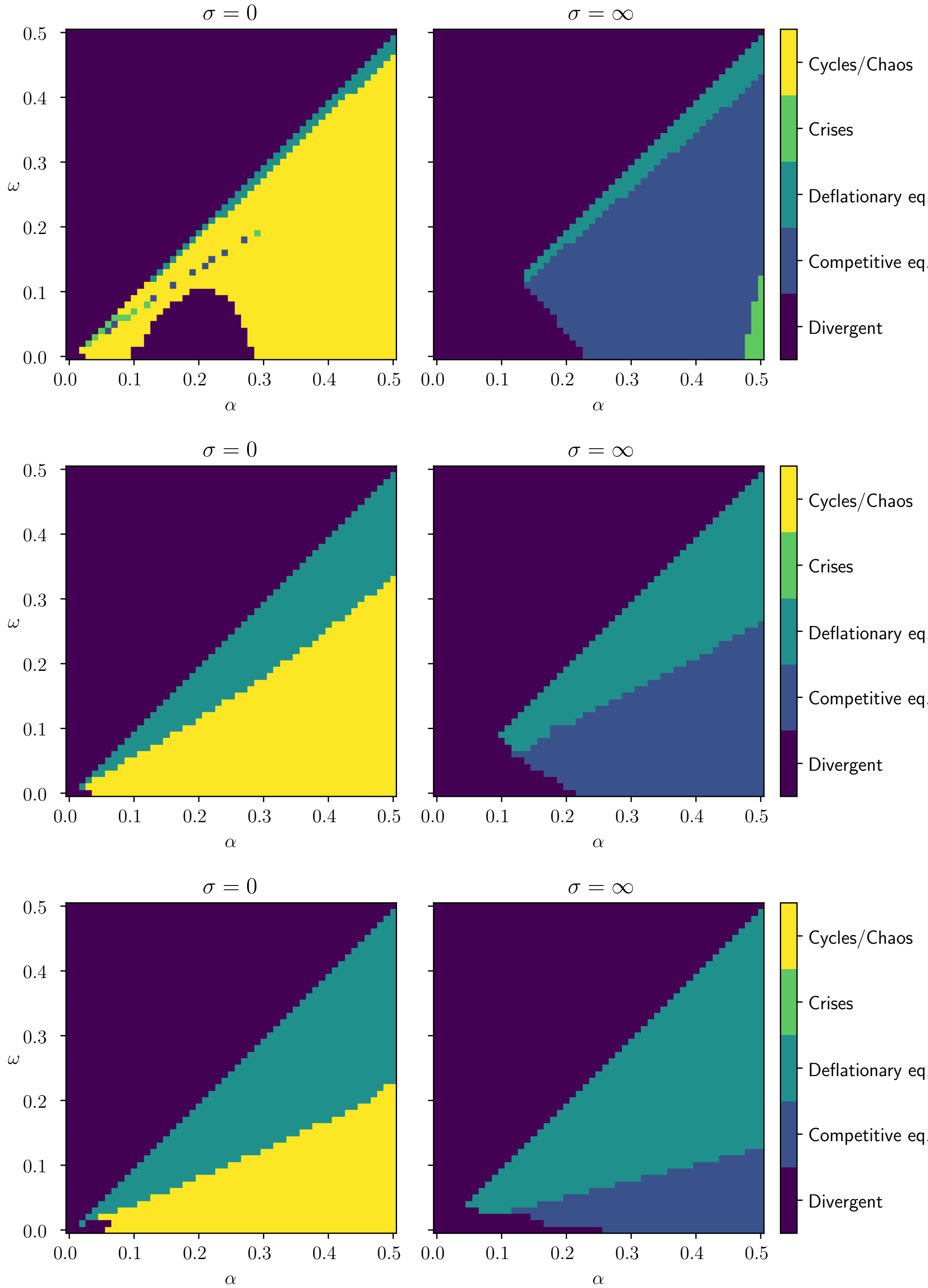

FIG. 10: Phase diagrams for non-perishable ($\sigma = 0$, left column) and immediately perishable ($\sigma = \infty$, right column) goods and for different values of ε (top: $\varepsilon = 100$, middle: $\varepsilon = 10$, bottom: $\varepsilon = 1$).

lead to simple relaxation towards equilibrium (deflationary/competitive) or to a divergence; on the other hand, non-perishable goods lead to oscillating, volatile economies. Intuitively, if firm i has a stock I_{ik} of good k, it will decrease its demand to firm k, leading to a decrease of its production. This lasts until all stocks are exhausted. A phase of booming demands and increase in production follows, firms' stocks begin to pile up again and the economy enters another cycle. This is similar to the well-known "bull-whip effect" [52], where inventories are known to lead to instability effects – these instabilities do indeed disappear completely when $\sigma = \infty$ (right column of Fig. 10).

VI. SUMMARY & CONCLUSION

Let us first summarize the main messages of this paper. We started from the observation made in [28] that generic input-output network models cannot reach an competitive equilibrium state when productivity is too low, or connectivity too high, or substitutability too low. This begs the question: what happens to the economy in such cases?

We argued that the answer to such a question is necessarily of dynamical nature, and demands an extension of the classical equilibrium framework (based on market-clearing and zero-profits requirements) to out-of-equilibrium equations of motion, that aim to describe *how* imbalances regress in time and *how fast* equilibrium is reached – if it is reached at all.

We first proposed what we called a “naive” model, based on the idea that forces driving the economy back to equilibrium are linear in the imbalances (profits and supply/demand imbalances). This leads to interesting non-linear differential equations for prices and productions which predict, among other things, that the equilibration time diverges as the network economy approaches the instability point at which competitive equilibrium is no longer realisable. We argued that this long time scale also leads to excess volatility, as the impact of exogenous shocks accumulates in the system.

We then pointed out that the naive model is numerically unstable as soon as perturbations away from equilibrium are not very small. This instability was traced back to the fact that the model does not correctly factor in physical constraints: excess demand cannot be satisfied, excess supply must be stored, consumption can only start after goods are produced, wages can only be spent after being paid, etc. Accounting for all these constraints within a consistent model considerably complexifies the resulting equations, but leads to a numerically stable model which can be used to explore a large variety of possible dynamical behaviour, even far from the competitive equilibrium. In fact, the model remains well-behaved even in the region of parameters where equilibrium is un-realisable.

A preliminary investigation of the full model leads to rich phase diagrams, which reveal that

- The competitive equilibrium attracts the dynamics only in a restricted range of parameters – the speed at which firms adapt to imbalances must neither be too slow nor too fast.
- When the adaptation speed is too large, coordination breaks down and the economy enters a phase with periodic or chaotic business cycles of purely endogenous origin, as was also reported in [31].
- Interestingly, other types of equilibria exist, with a negative inflation but with stationary real prices and production different from those pertaining to the competitive equilibrium. In particular, markets – including the job market – do not clear in such situations: labour supply is always larger than labour demand. For inflationary equilibria to appear (where labour demand is larger than labor supply), we need to introduce precautionary savings and interest rates.
- Finally, close to the boundaries between these phases one often observes a regime of intermittent crises, with long periods of quasi-equilibrium interrupted by bursts of inflation.

Hence, our model suggests two distinct routes to excess volatility (or “large business cycles”): purely *endogenous* cycles, resulting from non-linearities and feedback, or persistence and amplification of *exogenous shocks*, governed by the proximity of an instability point that leads to long relaxation times.

It should also be borne in mind that many relevant features of the real economy are left out of the present version of the model. In particular, while firms are allowed to make losses, we have not accounted to the cost of credit that this entails, nor have we introduced a bankruptcy mechanism when firms go too deep into debt. This would require moving from a static network of firms, as considered throughout this work, to a dynamically evolving network that rewires as some firms go bankrupt and others are created. In fact, another motivation for moving from a static framework to a dynamic model is to be able to describe possible *cascades of bankruptcies* mediated by the input-output network, much as cascades of defaults can occur in banking networks.

The household sector also needs to be better described, moving away from the representative household assumption and introducing wage inequalities, confidence effects and debt.

In fact, our dynamical model can be seen as a hybrid between traditional economic models (describing the static problem) and Agent Based Models, where extra reasonable but *ad hoc* rules are implemented to account for out-of-equilibrium, dynamical aspects. As we have shown, in some swath of parameters, the classical competitive equilibrium is reached. If reached fast enough, the "adiabatic" assumption used in most classical descriptions will hold, whereas when the equilibration time is long (or even infinite) new phenomena appear. We hope that this possibility of recovering standard results in some limiting cases will make the ABM approach more palatable to economists, and at the same time elicit the inherent limits of general equilibrium ideas. Conversely, including firm network effects in ABMs like Mark-0 [26, 46] along the lines of the present model is a promising path.

An interesting feature of our approach is the possibility of using highly disaggregated data on individual firms and prices (for example through the "Billion Price Project" [53]) to calibrate the model and, hopefully, use it as a powerful descriptive and predictive tool. We look forward to working in that direction in the near future.

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NOTATIONS

In this section, we summarize all the key notations that are used throughout the paper.

Production function and networks

- ⋆ q is the elasticity of substitution between inputs. The case $q = 0$ corresponds to a Leontief production function where inputs are not substitutable to one another whereas $q = \infty$ corresponds to a Cobb-Douglas production function where inputs are fully substitutable.
- ⋆ b is the return-to-scale parameter.
- ⋆ $\mathbf{J} \in \mathcal{M}_{N,N+1}(\mathbb{R})$ is the input-output matrix. Its entries J_{ij} denote the amount of inputs made by j needed by i to produce one unit of its good, and therefore defines a weighed adjacency matrix and an interaction network. Conventionally, the input $j = 0$ corresponds to labour and we use the notation $J_{i0} = V_i$.
- ⋆ $\mathbf{a} \in \mathcal{M}_{N,N+1}(\mathbb{R})$ is the substitution matrix. Its entries a_{ij} and a_{ik} indicate the ease with which firm i can replace an input k with another input j. For example, $a_{i0} = 0$ determines how labour may substitute other inputs.
- ⋆ $\mathbf{\Lambda} \in \mathcal{M}_{N,N+1}(\mathbb{R}) = \mathbf{a}^{q\zeta} \circ \mathbf{J}^{\zeta}$ is the aggregate matrix for the Constant-Elasticity of Substitution production function.
- ⋆ $\boldsymbol{\mathcal{M}} = \mathbf{\Delta}(z_i) - \mathbf{\Lambda}$ is the network matrix with the productivity factors of the firms on the diagonal. We implicitly cross out the first column of $\mathbf{\Lambda}$.
- ⋆ ε is the smallest eigenvalue of the network matrix.

Firms

- ⋆ N is the number of firms.
- ⋆ z_i is the productivity factor of firm i.
- ⋆ α is the log-elasticity of prices' growth rates against production surplus.
- ⋆ α' is the log-elasticity of prices' growth rates against profits.

- ⋆ β is the log-elasticity of productions' growth rates against profits.
- ⋆ β' is the log-elasticity of productions' growth rates against production surplus.
- ⋆ ω is the log-elasticity of wage's growth rate against labor market tensions.
- ⋆ σ_i are the depreciation parameters of goods.
- ⋆ $p_i(t) \in \mathbb{R}^N$ is the price of good i time t.
- ⋆ $p_0(t)$ is the common wage used to pay the household at time t.
- ⋆ $\pi_i(t) := z_i\gamma_i(t)$ is the production of firm i at time t along with the corresponding production levels $\gamma_i(t)$ at time t.
- ⋆ $\widehat{\pi}_i(t) := z_i\widehat{\gamma}_i(t)$ is the targeted productions by firm i at time t along with the corresponding targeted production level $\widehat{\gamma}_i(t)$ at time t.
- ⋆ $I_{ij}(t) \in \mathcal{M}_N(\mathbb{R})$ are the inventories. The diagonal terms $I_{ii}(t)$ correspond to the inventory of a firms own good whereas $I_{ij}(t)$ correspond to inventories of a firms' inputs.
- ⋆ $\mathscr{G}_i(t)$, $\mathscr{L}_i(t)$, $\mathscr{S}_i(t)$ and $\mathscr{D}_i(t)$ correspond respectively to the proceeds of sales ("gains"), the production costs ("losses"), the supplies and the demand for each firm at time t.
- ⋆ $\mathscr{P}_i(t) := \mathscr{G}_i(t) - \mathscr{L}_i(t)$ are each firm's realized profits at time t.
- ⋆ $\mathscr{E}_i(t) := \mathscr{S}_i(t) - \mathscr{D}_i(t)$ are each firm's production surplus at time t.
- ⋆ $\widehat{Q}_{ij}(t)$ is the quantity of good j that minimizes the costs of firm i given a certain production target and a production function. $\widehat{Q}_{i0}(t) := \widehat{\ell}_i(t)$ corresponds to the optimal amount of work required.
- ⋆ $Q^{\mathrm{d}}_{ij}(t)$ is the quantity of input j that is demanded by firm i to firm j. $Q^{\mathrm{d}}_{i0}(t) := \ell^{\mathrm{d}}_i(t)$ corresponds to the demanded amount of work.
- ⋆ $Q_{ij}(t)$ is the quantity of input j that is effectively exchanged. $Q_{i0}(t) := \ell_i(t)$ corresponds to the amount of work the household is hired to do for firm i.
- ⋆ $Q^{\mathrm{a}}_{ij}(t)$ is the quantity of input j that is available for production. $Q^{\mathrm{a}}_{i0}(t) := \ell^{\mathrm{a}}_i(t)$ corresponds to the available workforce for production.
- ⋆ $Q^{\mathrm{u}}_{ij}(t)$ is the quantity of input j that effectively used for production. $Q^{\mathrm{u}}_{i0}(t) := \ell^{\mathrm{u}}_i(t)$ corresponds to the available workforce for production.
- ⋆ λ is a behavioral parameter determining how firms forecast their future exchanges.

Household

- ⋆ θ_i is the consumption preference of the household for good i.
- ⋆ $\bar{\theta} = \sum_i \theta_i$.
- ⋆ L_0 is the nominal number of hours that the household is willing to work.
- ⋆ Γ is the aversion to work parameter.
- ⋆ φ is the convexity-to-work parameter.
- ⋆ ω' is a consumption confidence parameter.
- ⋆ $\mathcal{U}(t)$ is the utility of the household at time t.
- ⋆ $L^{\mathrm{s}}(t)$ is the available supply of work at time t.
- ⋆ $L^{\mathrm{d}}(t)$ is the total demand for work at time t.
- ⋆ $L(t)$ is the actual amount of work done at time t.
- ⋆ $C^{\mathrm{d}}_i(t)$ is the demanded consumption at time t.
- ⋆ $C^{\mathrm{r}}_i(t)$ is the realized consumption at time t.
- ⋆ $B(t)$ is the budget at time t.
- ⋆ $S(t)$ are the savings at time t.

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Appendix A: General Equilibrium Conditions

In this appendix, we show the computations that lead to the equilibrium equations on prices and production levels in the case of a general CES production function and a non-constant return-to-scale b.

1. Case $q < +\infty$

We first enforce the market clearing condition

$$z_i \gamma_{\text{eq},i} = \sum_{j=1}^{N} Q_{\text{eq},ji} + C_{\text{eq},i}, \tag{A.1}$$

and inject it into the zero-profit condition using (II.3). We can deduce a nicer expression for the quantity $p^{\text{net}}_{\text{eq},i} = \sum_{j=0}^{N} \Lambda_{ij} p^{\zeta}_{\text{eq},j}$ at equilibrium:

$$\begin{aligned} z_i p_{\text{eq},i} \gamma_{\text{eq},i} = \sum_{j=1}^{N} \Lambda^{a}_{ij} p^{\zeta}_{\text{eq},j} \left(p^{\text{net}}_{\text{eq},i}\right)^{q} \gamma^{1/b}_{\text{eq},i} &\iff z_i p_{\text{eq},i} \gamma^{\frac{b-1}{b}}_{\text{eq},i} = \left(p^{\text{net}}_{\text{eq},i}\right)^{q} \sum_{j} \Lambda_{ij} p^{\zeta}_{\text{eq},j} \\ &\iff \left(p^{\text{net}}_{\text{eq},i}\right)^{q+1} = z_i p_{\text{eq},i} \gamma^{\frac{b-1}{b}}_{\text{eq},i} \\ &\iff p^{\text{net}}_{\text{eq},i} = \left(z_i p_{\text{eq},i} \gamma^{\frac{b-1}{b}}_{\text{eq},i}\right)^{\zeta}, \end{aligned}$$

and therefore a nicer expression for the exchanged quantities

$$Q^{eq}_{ij} = \Lambda_{ij} p^{-q\zeta}_{\text{eq},j} z_i^{q\zeta} p^{q\zeta}_{\text{eq},i} \gamma^{\frac{\zeta(b-1)+1}{b}}_{\text{eq},i} = z_i^{q\zeta} \Lambda^{a}_{ij} \left(\frac{p_{\text{eq},i}}{p_{\text{eq},j}}\right)^{q\zeta} \gamma^{\zeta\frac{bq+1}{b}}_{\text{eq},i}. \tag{A.2}$$

Using the null budget condition we can retrieve the equilibrium consumption

$$C^{eq}_i = \frac{\theta_i}{\bar{\theta}^{\frac{\varphi}{1+\varphi}} \Gamma^{\frac{1}{1+\varphi}}} \frac{L_0}{p_{\text{eq},i}} := \frac{\kappa_i}{p_{\text{eq},i}}, \tag{A.3}$$

so that we have every ingredients to get closed form equations on prices and production levels. We express (A.2) and (A.3) back into the the zero profit condition to retrieve the first equilibrium equation:

$$\begin{aligned} &\forall i,\ z_i p_{\text{eq},i} \gamma_{\text{eq},i} - \sum_{j=1}^{N} p_{\text{eq},j} z_i^{q\zeta} \Lambda_{ij} \left(\frac{p_{\text{eq},i}}{p_{\text{eq},j}}\right)^{q\zeta} \gamma^{\zeta\frac{bq+1}{b}}_{\text{eq},i} = z_i^{q\zeta} \Lambda^{a}_{i0} p^{q\zeta}_{\text{eq},i} \gamma^{\zeta\frac{bq+1}{b}}_{\text{eq},i} \\ \iff &\forall i,\ z_i^{\zeta} p^{\zeta}_{\text{eq},i} \gamma^{\zeta\frac{b-1}{b}}_{\text{eq},i} - \sum_{j=1}^{N} \Lambda_{ij} p^{\zeta}_{\text{eq},j} = \Lambda_{i0} \\ \iff &\forall i,\ z_i^{\zeta} p^{\zeta}_{\text{eq},i} - \sum_{j=1}^{N} \Lambda_{ij} p^{\zeta}_{\text{eq},j} = \Lambda_{i0} + z_i^{\zeta} p^{\zeta}_{\text{eq},i} \left(1 - \gamma^{\zeta\frac{b-1}{b}}_{\text{eq},i}\right) \\ \iff &\boldsymbol{\mathcal{M}} {\mathbf{p}_{\text{eq}}}^{\zeta} = \mathbf{V} + \mathbf{z}^{\zeta} \circ {\mathbf{p}_{\text{eq}}}^{\zeta} \circ \left(1 - {\boldsymbol{\gamma}_{\text{eq}}}^{\zeta\frac{b-1}{b}}\right), \end{aligned}$$

and then in the market clearing condition to retrieve the second equilibrium equation:

$$
\begin{aligned}
&\forall i,\ z_i\gamma_{\mathrm{eq},i} - \sum_{j=1}^{N} z_j^{q\zeta}\Lambda_{ji}\left(\frac{p_{\mathrm{eq},j}}{p_{\mathrm{eq},i}}\right)^{q\zeta}\gamma_{\mathrm{eq},j}^{\zeta\frac{bq+1}{b}} = \frac{\kappa_i}{p_{\mathrm{eq},i}}\\
\iff &\forall i,\ z_i\gamma_{\mathrm{eq},i}p_{\mathrm{eq},i}^{q\zeta} - \sum_{j=1}^{N} z_j^{q\zeta}\Lambda_{ji}p_{\mathrm{eq},j}^{q\zeta}\gamma_{\mathrm{eq},j}^{\zeta\frac{bq+1}{b}} = \frac{\kappa_i}{p_{\mathrm{eq},i}^{\zeta}}\\
\iff &\forall i,\ z_i^{\zeta}\gamma_{\mathrm{eq},i}z_i^{q\zeta}p_{\mathrm{eq},i}^{q\zeta} - \sum_{j=1}^{N} \Lambda_{ji}z_j^{q\zeta}p_{\mathrm{eq},j}^{q\zeta}\gamma_{\mathrm{eq},j}^{\zeta\frac{bq+1}{b}} = \frac{\kappa_i}{p_{\mathrm{eq},i}^{\zeta}}\\
\iff &\forall i,\ z_i^{\zeta}\gamma_{\mathrm{eq},i}^{\zeta\frac{bq+1}{b}}z_i^{q\zeta}p_{\mathrm{eq},i}^{q\zeta} - \sum_{j=1}^{N} \Lambda_{ji}z_j^{q\zeta}p_{\mathrm{eq},j}^{q\zeta}\gamma_{\mathrm{eq},j}^{\zeta\frac{bq+1}{b}} = \frac{\kappa_i}{p_{\mathrm{eq},i}^{\zeta}} + z_i p_{\mathrm{eq},i}^{q\zeta}\gamma_{\mathrm{eq},i}^{\zeta\frac{bq+1}{b}}\left(1-\boldsymbol{\gamma}_{\mathrm{eq}}^{\zeta\frac{b-1}{b}}\right)\\
\iff &\boldsymbol{\mathcal{M}}^t\boldsymbol{\Delta}\left(\mathbf{z}^{q\zeta}\mathbf{p}_{\mathrm{eq}}{}^{q\zeta}\right)\boldsymbol{\gamma}_{\mathrm{eq}}{}^{\zeta\frac{bq+1}{b}} = \frac{\boldsymbol{\kappa}}{\mathbf{p}_{\mathrm{eq}}{}^{\zeta}} + \mathbf{z}\circ\mathbf{p}_{\mathrm{eq}}{}^{q\zeta}\circ\boldsymbol{\gamma}_{\mathrm{eq}}{}^{\zeta\frac{bq+1}{b}}\left(1-\boldsymbol{\gamma}_{\mathrm{eq}}{}^{\zeta\frac{b-1}{b}}\right).
\end{aligned}
$$

In the case where $q\to 0^+$ and $b=1$, one can check that (II.6) is retrieved.

2. Case $q=+\infty$

To retrieve the equations in the case $q=+\infty$, we need to take this limit in (II.3). It yields

$$
\begin{aligned}
\widehat{Q}_{il} &= a_{il}^{q\zeta}J_{il}^{\zeta}p_l^{-q\zeta}\left(\sum_{j=0}^{N} a_{ij}^{q\zeta}J_{ij}^{\zeta}p_j^{\zeta}\right)^{q}\hat{\gamma}_i^{1/b}\\
&= a_{il}^{q\zeta}J_{il}^{\zeta}p_l^{-q\zeta}\left(\sum_{\substack{j=0\\J_{ij}\neq 0}}^{N} a_{ij}^{q\zeta}J_{ij}^{\zeta}p_j^{\zeta}\right)^{q}\hat{\gamma}_i^{1/b}\\
&\underset{q\to+\infty}{\approx} a_{il}p_l\hat{\gamma}_i^{1/b}\exp\left\{q\log\sum_{\substack{j=0\\J_{ij}\neq 0}}^{N} a_{ij}\exp\zeta\log\left[\frac{J_{ij}}{a_{ij}}p_j\right]\right\}\\
&\underset{q\to+\infty}{\approx} a_{il}p_l^{-1}\hat{\gamma}_i^{1/b}\exp\left\{q\log\sum_{\substack{j=0\\J_{ij}\neq 0}}^{N} a_{ij} + \zeta\sum_{\substack{j=0\\J_{ij}\neq 0}}^{N}\log\left[\frac{J_{ij}}{a_{ij}}p_j\right]\right\}\\
&\underset{q\to+\infty}{\approx} a_{il}p_l^{-1}\hat{\gamma}_i^{1/b}\exp\left\{q\log\left(1+\zeta\sum_{\substack{j=0\\J_{ij}\neq 0}}^{N}\log\left[\frac{J_{ij}}{a_{ij}}p_j\right]\right)\right\}\\
&\underset{q\to+\infty}{\approx} a_{il}p_l^{-1}\hat{\gamma}_i^{1/b}\exp\left\{q\zeta\sum_{\substack{j=0\\J_{ij}\neq 0}}^{N}\log\left[\frac{J_{ij}}{a_{ij}}p_j\right]\right\}\\
&\underset{q\to+\infty}{\approx} a_{il}p_l^{-1}\hat{\gamma}_i^{1/b}\prod_{\substack{j=0\\J_{ij}\neq 0}}^{N}\left(\frac{J_{ij}}{a_{ij}}p_j\right).
\end{aligned}
$$

We can then express the quantity $z_i\gamma_{\text{eq},i}^{\frac{b-1}{b}}p_{\text{eq},i}$ through the zero profit condition as

$$z_i\gamma_{\text{eq},i}^{\frac{b-1}{b}}p_{\text{eq},i} = \prod_{\substack{j=0\\J_{ij}\neq 0}}^{N}\left(\frac{J_{ij}}{a_{ij}}p_{\text{eq},j}\right). \tag{A.4}$$

Using the market clearing condition, we can get the first equilibrium equation in the Cobb-Douglas case:

$$\begin{aligned}
&\forall i,\ z_i\gamma_{\text{eq},i} = \frac{\kappa_i}{p_{\text{eq},i}} + \sum_j a_{ji}p_{\text{eq},i}^{-1}\gamma_{\text{eq},j}^{1/b}\prod_{\substack{j=0\\J_{ij}\neq 0}}^{N}\left(\frac{J_{ij}}{a_{ij}}p_{\text{eq},j}\right)\\
\iff\ &\forall i,\ z_i\gamma_{\text{eq},i}p_{\text{eq},i} = \kappa_i + \sum_j a_{ji}z_j\gamma_{\text{eq},j}p_{\text{eq},j}\\
\iff\ &\left(\mathbf{I}_N - \mathbf{a}^t\right)\mathbf{z}\circ\boldsymbol{\gamma}_{\text{eq}}\circ\mathbf{p}_{\text{eq}} = \boldsymbol{\kappa}\\
\iff\ &\mathbf{z}\circ\boldsymbol{\gamma}_{\text{eq}}\circ\mathbf{p}_{\text{eq}} = \left(\mathbf{I}_N - \mathbf{a}^t\right)^{-1}\boldsymbol{\kappa}.
\end{aligned}$$

To get the second equation, we inject the previous into (A.4) and take the logarithm. It reads

$$\begin{aligned}
&\forall i,\ \log z_i\gamma_{\text{eq},i}p_{\text{eq},i} - \frac{1}{b}\log\gamma_{\text{eq},i} = \sum_{\substack{l=1\\J_{il}\neq 0}}^{N} a_{il}\log\frac{J_{il}}{a_{il}} + \sum_{\substack{l=1\\J_{il}\neq 0}}^{N} a_{il}\log p_{\text{eq},i}\\
\iff\ &\forall i,\ \frac{b-1}{b}\log\left[\left(\mathbf{I}_N - \mathbf{a}^t\right)^{-1}\boldsymbol{\kappa}\right]_i + \frac{1}{b}\log p_{\text{eq},i} + \frac{1}{b}\log z_i = \sum_{\substack{l=1\\J_{il}\neq 0}}^{N} a_{il}\log\frac{J_{il}}{a_{il}} + \sum_{\substack{l=1\\J_{il}\neq 0}}^{N} a_{il}\log p_{\text{eq},i}\\
\iff\ &\left(\frac{1}{b}\mathbf{I}_N - \mathbf{a}\right)\log\mathbf{p}_{\text{eq}} = \frac{1-b}{b}\log\left(\mathbf{I}_N - \mathbf{a}^t\right)^{-1}\boldsymbol{\kappa} - \frac{1}{b}\log\mathbf{z} + \mathbf{h}.
\end{aligned}$$

where $h_i = \sum_{\substack{l=1\\J_{il}\neq 0}}^{N} a_{il}\log\frac{J_{il}}{a_{il}}$.

Appendix B: Relaxation Time for the Naive Model

The non-linear dynamics of the naive model are given by (III.4). In this Appendix, we derive the relaxation time of the system in various limits.

1. Linearisation of the dynamics

To linearise the system ,we write $p_i(t) = p_{\text{eq},i} + \delta p_i(t)$ and $\gamma_i(t) = \gamma_{\text{eq},i} + \delta\gamma_i(t)$ and inject these expressions into (III.4). After a few computations, we establish the following linear equation in the variable $\mathbf{U}(t) = (\delta\mathbf{p}(t), \delta\boldsymbol{\gamma}(t))^t$ to first order:

$$\frac{d\mathbf{U}}{dt} = \begin{pmatrix}\mathbf{D}_1 & \mathbf{D}_2\\ \mathbf{D}_3 & \mathbf{D}_4\end{pmatrix}\mathbf{U}(t) := \mathbb{D}\mathbf{U}(t), \tag{B.1}$$

where the different blocks of the matrix are

$$\begin{aligned}
\mathbf{D}_1 &= -\alpha\mu\boldsymbol{\Delta}\left(\frac{\theta_i}{z_i\gamma_{\text{eq},i}p_{\text{eq},i}}\right) - \alpha'\boldsymbol{\Delta}\left(z_i^{-1}\right)\boldsymbol{\mathcal{M}} \qquad & \mathbf{D}_2 &= -\alpha\boldsymbol{\Delta}\left(\frac{p_{\text{eq},i}}{z_i\gamma_{\text{eq},i}}\right)\boldsymbol{\mathcal{M}}^t\\
\mathbf{D}_3 &= \beta\boldsymbol{\Delta}\left(\frac{\gamma_{\text{eq},i}}{z_ip_{\text{eq},i}}\right)\boldsymbol{\mathcal{M}} - \beta'\mu\boldsymbol{\Delta}\left(\frac{\theta_i}{z_ip_{\text{eq},i}^2}\right) & \mathbf{D}_4 &= -\beta'\boldsymbol{\Delta}\left(z_i^{-1}\right)\boldsymbol{\mathcal{M}}^t.
\end{aligned} \tag{B.2}$$

2. Relaxation time in the high productivity regime

In this section, we assume that the productivity factors are large enough to ignore interactions between firms. In this regime, firms are efficient enough so that the actual amount of inputs does not matter in the final production. In this limit, we can give approximate expressions for the equilibrium prices and productions

$$p_{\mathrm{eq},i} = \frac{V_i}{z_i} \tag{B.3a}$$

$$\gamma_{\mathrm{eq},i} = \frac{\mu\theta_i}{V_i}. \tag{B.3b}$$

Similarly, we approximate each block of the stability matrix:

$$\begin{aligned} \mathbf{D}_1 &\underset{z_i\to\infty}{\approx} -(\alpha+\alpha')\mathbf{I}_N & \mathbf{D}_2 &\underset{z_i\to\infty}{\approx} -\alpha\boldsymbol{\Delta}\left(\frac{V_i^2}{z_i\mu\theta_i}\right) \\ \mathbf{D}_3 &\underset{z_i\to\infty}{\approx} (\beta-\beta')\boldsymbol{\Delta}\left(\frac{z_i\mu\theta_i}{V_i^2}\right) & \mathbf{D}_4 &\underset{z_i\to\infty}{\approx} -\beta'\mathbf{I}_N, \end{aligned} \tag{B.4}$$

and deduce the spectrum of the $\mathbf{D}$ by computing its characteristic polynomial and setting it to 0:

$$\begin{aligned} \det\left(\sigma\mathbf{I}_{2N} - \mathbb{D}\right) &= \begin{vmatrix} \sigma\mathbf{I}_N - \mathbf{D}_1 & -\mathbf{D}_2 \\ -\mathbf{D}_3 & \sigma\mathbf{I}_N - \mathbf{D}_4 \end{vmatrix} \\ &\underset{z_i\to\infty}{\approx} \det\left(\left(\sigma+\alpha+\alpha'\right)\left(\sigma+\beta'\right)\mathbf{I}_N + \alpha(\beta-\beta')\mathbf{I}_N\right) \\ &= \left(\sigma^2 + \sigma(\alpha+\alpha'+\beta') + \alpha\beta + \alpha'\beta'\right)^N \\ &= 0. \end{aligned}$$

Solving this equation yields two eigenvalues $\sigma_\pm$, both with degeneracy N, that read

$$\sigma_\pm = \frac{1}{2}\times\begin{cases} -\alpha'-\beta'-\alpha\pm\sqrt{(\alpha'+\beta'+\alpha)^2 - 4(\alpha\beta+\alpha'\beta')} & \text{if} \quad (\alpha'+\beta'+\alpha)^2 > 4(\alpha\beta+\alpha'\beta') \\ -\alpha'-\beta'-\alpha\pm i\sqrt{4(\alpha\beta+\alpha'\beta') - (\alpha'+\beta'+\alpha)^2} & \text{if} \quad (\alpha'+\beta'+\alpha)^2 < 4(\alpha\beta+\alpha'\beta') \end{cases}. \tag{B.5}$$

This in turn lets us deduce the relaxation time:

$$\tau_{relax} = 2\times\begin{cases} (\alpha'+\beta'+\alpha-\sqrt{(\alpha'+\beta'+\alpha)^2 - 4(\alpha\beta+\alpha'\beta')})^{-1} & \text{if} \quad (\alpha'+\beta'+\alpha)^2 > 4(\alpha\beta+\alpha'\beta') \\ (\alpha'+\beta'+\alpha)^{-1} & \text{if} \quad (\alpha'+\beta'+\alpha)^2 \geq 4(\alpha\beta+\alpha'\beta') \end{cases}. \tag{B.6}$$

3. Perturbation expansion in ε for $\mathbb{D}$

Studying the behaviour of $\mathbb{D}$ as $\varepsilon \to 0^+$ requires understanding the behaviour of $\left(\boldsymbol{\mathcal{M}}, \mathbf{p}_{eq}, \boldsymbol{\gamma}_{eq}\right)$ in that limit. We now introduce the matrix $\widetilde{\mathbf{J}} = \boldsymbol{\Delta}\left(z_{\max} - z_i\right) + \mathbf{J}$ and denote by ρ_ν (resp. $|r_\nu\rangle$, $\langle\ell_\nu|$)[17] its eigenvalues (resp. right/left eigenvectors) ordered by their real parts. The Perron-Froebenius theorem implies that the top eigenvalue ρ_N is real, simple and associated to a full and positive eigenvector. We next use the following spectral representation of the matrix $\boldsymbol{\mathcal{M}}$:

$$\boldsymbol{\mathcal{M}} = \left(\rho_N\mathbf{I}_N - \widetilde{\mathbf{J}}\right) + \varepsilon\mathbf{I}_N \tag{B.7}$$

$$\begin{aligned} \boldsymbol{\mathcal{M}}^{-1} &= \frac{1}{\varepsilon}\left|r_N\right\rangle\left\langle\ell_N\right| + \sum_{\nu=1}^{N-1}\frac{1}{\rho_N-\rho_\nu+\varepsilon}\left|r_\nu\right\rangle\left\langle\ell_\nu\right| \\ &= \frac{1}{\varepsilon}\left|r_N\right\rangle\left\langle\ell_N\right| + \sum_{k=0}^{\infty}(-\varepsilon)^k\sum_{\nu=1}^{N-1}\frac{1}{(\rho_N-\rho_\nu)^{k+1}}\left|r_\nu\right\rangle\left\langle\ell_\nu\right|, \end{aligned} \tag{B.8}$$

[17] We use here Dirac bra-ket notation, where $|v\rangle$ represents a column vector and $\langle v|$ a row vector.

which lets us express the equilibrium prices and outputs as well as $\mathbb{D}$. We also use the notation $\boldsymbol{\mathcal{M}}_0$ to refer to the network matrix when $\varepsilon = 0$. This matrix is singular and verifies

$$\boldsymbol{\mathcal{M}}_0 \left| r_N \right\rangle = 0 \quad , \quad \boldsymbol{\mathcal{M}}_0^t \left| \ell_N \right\rangle = 0. \tag{B.9}$$

Expanding in ε and neglecting factors of order ε^4 and higher gives the following for the blocks of the stability matrix:

$$\begin{aligned} &\mathbf{D}_1 = \mathbf{D}_1^{(0)} + \varepsilon \mathbf{D}_1^{(1)} + \varepsilon^2 \mathbf{D}_1^{(2)} + \varepsilon^3 \mathbf{D}_1^{(3)} && \mathbf{D}_2 = \frac{1}{\varepsilon} \mathbf{D}_2^{(-1)} + \mathbf{D}_2^{(0)} + \varepsilon \mathbf{D}_2^{(1)} + \varepsilon^2 \mathbf{D}_2^{(2)} + \varepsilon^3 \mathbf{D}_2^{(3)} \\ &\mathbf{D}_3 = \varepsilon \mathbf{D}_3^{(1)} + \varepsilon^2 \mathbf{D}_3^{(2)} + \varepsilon^3 \mathbf{D}_3^{(3)} && \mathbf{D}_4 = \mathbf{D}_4^{(0)} + \varepsilon \mathbf{D}_4^{(1)} + \varepsilon^2 \mathbf{D}_4^{(2)} + \varepsilon^3 \mathbf{D}_4^{(3)}, \end{aligned} \tag{B.10}$$

where the exact definition of the perturbation terms $\mathbf{D}_i^{(l)}$ is given in the Appendix C. To ease computations and give closed-form results, we consider an undirected network (symmetric $\boldsymbol{\mathcal{M}}$) with homogeneous productivity factors. The qualitative results are however unchanged when considering more general networks. In this setting, the eigenvectors of $\boldsymbol{\mathcal{M}}$ are denoted by $|e_\nu\rangle$.

4. Marginal stability for $\varepsilon = 0$

Interestingly enough, although the upper-right block of $\mathbb{D}$ diverges as $\varepsilon \to 0$, its spectrum converges to a finite limit. To see this, we use the block determinant formula

$$\begin{vmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{vmatrix} = \det\left(\mathbf{AD} - \mathbf{BC}\right),$$

for same-size matrices, where the commutator $[\mathbf{C}, \mathbf{D}] = \mathbf{CD} - \mathbf{DC} = 0$. In our case, we need $[\mathbf{D}_3, \mathbf{D}_4] = 0$ which is true only in the limit $\varepsilon = 0$. We can then write:[18]

$$\begin{aligned} \det\left(\sigma \mathbf{I}_{2N} - \mathbb{D}\right) &\underset{\varepsilon\to 0}{\approx} \det\left(\left(\sigma \mathbf{I}_N - \mathbf{D}_1^{(0)}\right)\left(\sigma \mathbf{I}_N - \mathbf{D}_4^{(0)}\right) - \mathbf{D}_2^{(-1)} \mathbf{D}_3^{(1)}\right) \\ &= \det\left(\left(\sigma \mathbf{I}_N + \frac{\alpha'}{\rho_N} \boldsymbol{\mathcal{M}}_0\right)\left(\sigma \mathbf{I}_N + \frac{\beta'}{\rho_N} \boldsymbol{\mathcal{M}}_0\right) + \frac{\alpha\beta}{\rho_N^2} \boldsymbol{\mathcal{M}}_0^2\right) \\ &= \det\left(\sigma^2 \mathbf{I}_N + \sigma \frac{\alpha' + \beta'}{\rho_N} \boldsymbol{\mathcal{M}}_0 + \frac{\alpha\beta + \alpha'\beta'}{\rho_N^2} \boldsymbol{\mathcal{M}}_0^2\right) \\ &= \prod_{\nu=1}^{N} \left(\sigma^2 + \sigma \frac{\alpha' + \beta'}{\rho_N} (\rho_N - \rho_\nu) + \frac{\alpha\beta + \alpha'\beta'}{\rho_N^2} (\rho_N - \rho_\nu)^2\right) \\ &= \sigma^2 \prod_{\nu \neq N} \left(\sigma^2 + \sigma(\alpha' + \beta')\left(1 - \frac{\rho_\nu}{\rho_N}\right) + (\alpha\beta + \alpha'\beta')\left(1 - \frac{\rho_\nu}{\rho_N}\right)^2\right). \end{aligned}$$

Each factor in this product yields two eigenvalues:

- If $(\alpha' - \beta')^2 > 4\alpha\beta$ then

$$\sigma_\pm^\nu = \frac{1}{2}\left(-\alpha' - \beta' \pm \sqrt{(\alpha' + \beta')^2 - 4(\alpha\beta + \alpha'\beta')}\right)\left(1 - \frac{\rho_\nu}{\rho_N}\right), \tag{B.11}$$

- If $(\alpha' - \beta')^2 < 4\alpha\beta$ then

$$\sigma_\pm^\nu = \frac{1}{2}\left(-\alpha' - \beta' \pm i\sqrt{4(\alpha\beta + \alpha'\beta') - (\alpha' + \beta')^2}\right)\left(1 - \frac{\rho_\nu}{\rho_N}\right), \tag{B.12}$$

---

[18] We do not need to consider terms of order one in the commutator because $\left[\mathbf{D}_3^{(1)}, \mathbf{D}_4^{(0)}\right] = 0$, see Appendix C.

- If $(\alpha' - \beta')^2 = 4\alpha\beta$ then

$$\sigma_0^\nu = -\frac{\alpha' + \beta'}{2}\left(1 - \frac{\rho_\nu}{\rho_N}\right). \tag{B.13}$$

The trailing factor shows that 0 is an eigenvalue of $\mathbb{D}$ (for $\nu = N$), twice degenerated as $\varepsilon \to 0$. We deduce that the system exhibits marginal stability in this limit. Figure 11 shows the empirical distribution of eigenvalues of $\mathbb{D}$ and the corresponding theoretical predictions.

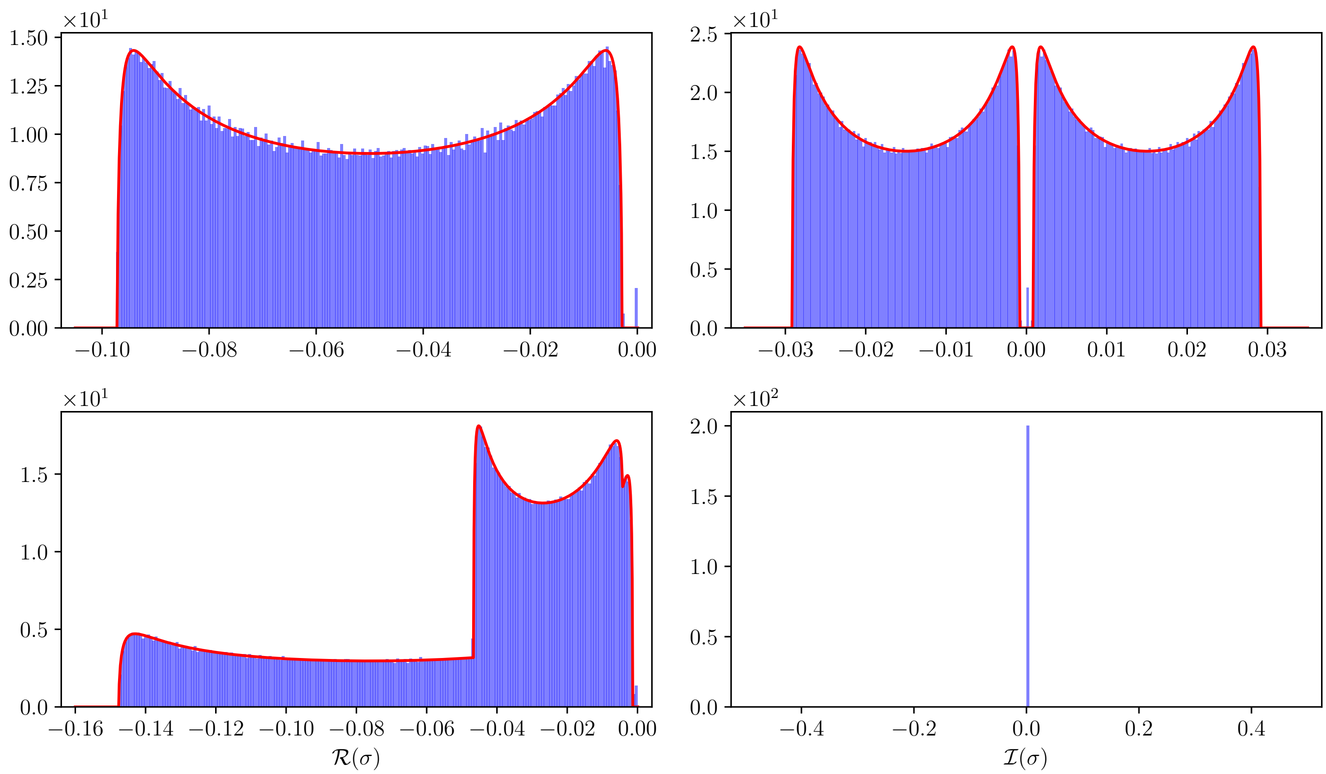

FIG. 11: Spectrum of the continuous stability matrix for $N = 1000$ firms on a d-regular undirected network with $d = 3$. Top: Case $(\alpha' - \beta')^2 < 4\alpha\beta$. Bottom: $(\alpha' - \beta')^2 > 4\alpha\beta$. The blue histogram gives the distribution of eigenvalues obtained through numerical diagonalizations of $\mathbb{D}$. The red line is the thermodynamic computation accounting for (B.12) and (B.11) using the McKay density for the eigenvalues of a random d-regular graph [43].

5. Relaxation time in the limit $\varepsilon \to 0$

We have thus far shown that our system exhibits marginal stability at $\varepsilon = 0$. We now prove that the relaxation time of the system behaves as $\tau_{relax} \sim \varepsilon^{-1}$. To this end, we use analytical perturbation theory as described in [54], which in our setting reduces to the ε-perturbation of the characteristic polynomial of $\mathbb{D}(0)$[19] as ε goes away from 0. This characteristic polynomial is given by

$$\chi(\sigma, 0) = \sigma^2 \prod_{\nu=1}^{N} \left(\sigma - \sigma_+^\nu\right)\left(\sigma - \sigma_-^\nu\right), \tag{B.14}$$

with $\sigma_\pm^\nu$ given in the previous section.

[19] We have done a slight abuse of notation, since $\mathbb{D}(0)$ is not formally defined because of the diverging upper right block.

We now try to find a perturbation of the σ^2 term to retrieve the perturbation on $\sigma_\pm^N = 0$. Using analytical perturbation theory, we see that $(\varepsilon, \sigma) = (0, 0)$ is a splitting point under the perturbation $\mathbb{D}(\varepsilon)$ ($\varepsilon = 0$ is a multiple point – since $\mathbb{D}$ has at least one multiple root for $\varepsilon = 0$ – and $\sigma_\pm^N = 0$ is a multiple root.) In this setting, $\sigma_\pm^N = 0$ splits under the perturbation $\mathbb{D}(\varepsilon)$ to give 2 perturbed eigenvalues. Henceforth, for small enough ε, the prime factor σ^2 of $\chi(\sigma, 0)$ is expressed as a second order polynomial whose coefficients depend on ε.

We may write

$$p_0(\sigma) := \sigma^2 \overset{\mathbb{D}(\varepsilon)}{\longrightarrow} p_0(\sigma, \varepsilon) := \sigma^2(1 + a_2^{(1)}\varepsilon + a_2^{(2)}\varepsilon^2 + \cdots) + \sigma(a_1^{(1)}\varepsilon + a_1^{(2)}\varepsilon^2 + \cdots) + a_0^{(1)}\varepsilon + a_0^{(2)}\varepsilon^2 + \cdots .$$

This expansion makes sure that $p_0(\sigma, \varepsilon) \underset{\varepsilon \to 0}{\longrightarrow} p_0(\sigma)$. Moreover, at least one of the $a_0^{(i)}$ is non-zero. Otherwise we would be able to factor out σ in $p_0(\sigma, \varepsilon)$, meaning that for small enough (but non zero) ε, $0 \in \mathrm{Sp}(\mathbb{D}(\varepsilon))$ which we know to be false because the system is stable for $\varepsilon > 0$.

Furthermore, we know that the splitting behaviour of $\sigma_\pm^N = 0$ is imposed, ensuring that the discriminant of $p_0(\sigma, \varepsilon)$ cannot vanish (leading to a multiple root), which yields another condition on the coefficients. Finally, since we are looking at complex roots in general, $p_0(\sigma, \varepsilon)$ will always factor into two irreducible and normalized polynomials of degree 1. This ensures that $\forall i \geq 1,\ a_2^{(i)} = 0$ and that $a_0^{(1)} = 0$.

This last point is not so straightforward and warrants an explanation. From [54], the Puiseux series for the perturbed eigenvalues $\sigma_{N\alpha}(\varepsilon)$ can be written as

$$\sigma_{N\alpha}(\varepsilon) = \sum_{x=1}^{\infty} b_{N\alpha x}\varepsilon^{x/g_{N\alpha}} \quad , \quad \alpha = 1, 2,$$

where $g_{N\alpha}$ is the degree of the polynomial from which the root $\sigma_{N\alpha}$ is extracted. In our setting $g_{N\alpha} = 1$ meaning that the first perturbation to $\sigma_{N\alpha}$ is of order ε. Now, we also know that $\sigma_{N\alpha}$ is obtained by solving the second order equation $p_0(\sigma, \varepsilon) = 0$. This means that both roots read

$$\sigma_{n\alpha} = o(\varepsilon) + \kappa_\alpha\sqrt{\Delta}.$$

We may now write Δ as

$$\Delta = o(\varepsilon^2) - 4a_0^{(1)}\varepsilon,$$

so that, if $a_0^{(1)} \neq 0$, the dominant term of $\sigma_{N\alpha}$ will be of order $o(\sqrt{\varepsilon})$ which contradicts the previous analysis.

Finally, we can attempting looking for a perturbation resembling

$$p_0(\sigma) := \sigma^2 \overset{\mathbb{D}(\varepsilon)}{\longrightarrow} p_0(\sigma, \varepsilon) := \sigma^2 + \sigma(a_1^{(1)}\varepsilon + a_1^{(2)}\varepsilon^2 + \cdots) + a_0^{(2)}\varepsilon^2 + \cdots .$$

To determine the different terms in this expansion, we re-use the determinant computation that we carried in the previous section, but keeping now terms up to order ε^2. This yields:

$$
\begin{aligned}
\det\left(\sigma\mathbf{I}_{2N}-\mathbb{D}\right) &= \begin{vmatrix}\sigma\mathbf{I}_N-\mathbf{D}_1 & -\mathbf{D}_2\\ -\mathbf{D}_3 & \sigma\mathbf{I}_N-\mathbf{D}_4\end{vmatrix}\\
&\underset{\varepsilon\to 0}{\approx} \det\left((\sigma\mathbf{I}_N-\mathbf{D}_1)(\sigma\mathbf{I}_N-\mathbf{D}_4)-\mathbf{D}_2\mathbf{D}_3\right)\\
&= \det\Bigg[\underbrace{\sigma^2\mathbf{I}_N-\sigma\left(\mathbf{D}_1^{(0)}+\mathbf{D}_4^{(0)}\right)+\mathbf{D}_1^{(0)}\mathbf{D}_4^{(0)}-\mathbf{D}_2^{(-1)}\mathbf{D}_3^{(1)}}_{\mathbf{\Sigma}^{(0)}(\sigma)}\\
&\qquad+\varepsilon\left(\underbrace{-\sigma(\mathbf{D}_1^{(1)}+\mathbf{D}_4^{(1)})+\mathbf{D}_1^{(0)}\mathbf{D}_4^{(1)}+\mathbf{D}_1^{(1)}\mathbf{D}_4^{(0)}-\mathbf{D}_2^{(-1)}\mathbf{D}_3^{(2)}-\mathbf{D}_2^{(0)}\mathbf{D}_3^{(1)}}_{\mathbf{\Sigma}^{(1)}(\sigma)}\right)\\
&\qquad+\varepsilon^2\left(\underbrace{-\sigma(\mathbf{D}_1^{(2)}+\mathbf{D}_4^{(2)})+\mathbf{D}_1^{(0)}\mathbf{D}_4^{(2)}+\mathbf{D}_1^{(1)}\mathbf{D}_4^{(1)}+\mathbf{D}_1^{(0)}\mathbf{D}_4^{(1)}-\mathbf{D}_2^{(-1)}\mathbf{D}_3^{(3)}-\mathbf{D}_2^{(0)}\mathbf{D}_3^{(2)}-\mathbf{D}_2^{(1)}\mathbf{D}_3^{(1)}}_{\mathbf{\Sigma}^{(2)}(\sigma)}\right)\Bigg]\\
&\underset{\varepsilon\to 0}{\approx} \det\mathbf{\Sigma}^{(0)}(\sigma)+\varepsilon\mathrm{Tr}\left(\mathrm{Com}\left(\mathbf{\Sigma}^{(0)}\right)^t\mathbf{\Sigma}^{(1)}(\sigma)\right)+\varepsilon^2\mathrm{Tr}\left(\mathrm{Com}\left(\mathbf{\Sigma}^{(0)}\right)^t(\sigma)\mathbf{\Sigma}^{(2)}(\sigma)\right)\\
&\qquad+\varepsilon^2\frac{\left(\mathrm{Tr}\left(\mathrm{Com}\left(\mathbf{\Sigma}^{(0)}\right)^t(\sigma)\mathbf{\Sigma}^{(1)}(\sigma)\right)\right)^2-\mathrm{Tr}\left(\left(\mathrm{Com}\left(\mathbf{\Sigma}^{(0)}\right)^t(\sigma)\mathbf{\Sigma}^{(1)}(\sigma)\right)^2\right)}{2\det\mathbf{\Sigma}^{(0)}(\sigma)}.
\end{aligned}
$$

The constant term $\det\mathbf{\Sigma}^{(0)}(\sigma)$ is the characteristic polynomial of $\mathbb{D}$ for $\varepsilon=0$ so that $\det\mathbf{\Sigma}^{(0)}(\sigma)=\chi(\sigma,0)$. Similarly, it is easy to prove that, for a diagonalizable matrix $\mathbf{A}$ with eigenvalues λ and associated eigenvector $|\lambda\rangle$, the matrix $\mathrm{Com}\ (\mathbf{A})$ can be diagonalized in the same basis and reads

$$
\mathrm{Com}\ (\mathbf{A})=\sum_{\lambda}\left(\prod_{\lambda'\neq\lambda}\lambda'\right)|\lambda\rangle\langle\lambda|. \tag{B.15}
$$

Using this lemma, we can write

$$
\mathrm{Com}\left(\mathbf{\Sigma}^{(0)}(\sigma)\right)=\left(\prod_{\nu\neq N}\left(\sigma-\sigma_+^{\nu}\right)\left(\sigma-\sigma_-^{\nu}\right)\right)|e_N\rangle\langle e_N|+\sum_{\nu\neq N}\left(\sigma^2\prod_{\mu\neq\nu,N}\left(\sigma-\sigma_+^{\mu}\right)\left(\sigma-\sigma_-^{\mu}\right)\right)|e_\nu\rangle\langle e_\nu|. \tag{B.16}
$$

We now develop each trace term onto the eigenbasis of $\mathrm{Com}\left(\mathbf{\Sigma}^{(0)}(\sigma)\right)$. From now on, we drop the σ dependencies of the $\mathbf{\Sigma}$ matrices but bear in mind that these matrices are polynomials of order one in σ. The first trace reads

$$
\begin{aligned}
\mathrm{Tr}\left(\mathrm{Com}\left(\mathbf{\Sigma}^{(0)}\right)^t\mathbf{\Sigma}^{(1)}\right)&=\left(\prod_{\nu\neq N}\left(\sigma-\sigma_+^{\nu}\right)\left(\sigma-\sigma_-^{\nu}\right)\right)\langle e_N|\,\mathbf{\Sigma}^{(1)}\,|e_N\rangle\\
&\quad+\sum_{\nu\neq N}\left(\sigma^2\prod_{\mu\neq\nu,N}\left(\sigma-\sigma_+^{\mu}\right)\left(\sigma-\sigma_-^{\mu}\right)\right)\langle e_\nu|\,\mathbf{\Sigma}^{(1)}\,|e_\nu\rangle.
\end{aligned}
$$

Only the first term is of interest for us and we can use the explicit forms of the blocks of $\mathbb{D}$ to find

$$
\begin{aligned}
\langle e_N|\,\mathbf{\Sigma}^{(1)}\,|e_N\rangle&=\sigma\langle e_N|\left(\mathbf{D}_1^{(1)}+\mathbf{D}_4^{(1)}\right)|e_N\rangle\\
&=-\frac{\sigma}{\rho_N}(\alpha+\alpha'+\beta').
\end{aligned}
$$

The same computation can be carried out for the second trace term,

$$\begin{aligned}\langle e_N|\,\boldsymbol{\Sigma}^{(2)}\,|e_N\rangle &= \sigma\,\langle e_N|\left(\mathbf{D}_1^{(2)}+\mathbf{D}_4^{(2)}\right)|e_N\rangle + \langle e_N|\,\mathbf{D}_1^{(1)}\mathbf{D}_4^{(1)}\,|e_N\rangle - \langle e_N|\,\mathbf{D}_2^{(0)}\mathbf{D}_3^{(2)}\,|e_N\rangle \\ &= \frac{\sigma}{\rho_N^2}(\alpha'+\beta') - \frac{\alpha'\beta'+\alpha\beta}{\rho_N^2} + \sigma\kappa,\end{aligned}$$

with $\kappa = \langle e_N|\,\mathbf{D}_1^{(2)}\,|e_N\rangle$ which we do not need to compute.

The square trace terms are very complicated, and we only sketch out their computation. The terms that could have entered in the perturbation of $p_0(\sigma)$ cancel out (these are sums of square terms). The terms that are rational fractions of polynomials (and could be pathological since we look for a polynomial perturbation) cancel out as well. The other terms do not enter the perturbation of $p_0(\sigma)$ and are non-pathological.

Finally the perturbation of $p_0(\sigma)$ resembles

$$p_0(\sigma) := \sigma^2 \xrightarrow{\mathbb{D}(\varepsilon)} p_0(\sigma,\varepsilon) \approx \sigma^2 + \sigma\left(\varepsilon\frac{\alpha+\alpha'+\beta'}{\rho_N} - \varepsilon^2\frac{\alpha'+\beta'}{\rho_N^2} - \varepsilon^2\kappa\right) + \varepsilon^2\frac{\alpha'\beta'+\alpha\beta}{\rho_N^2}.$$

We now write the discriminant of this polynomial at second order to get

$$\Delta(\varepsilon) = \frac{\varepsilon^2}{\rho_N^2}\left((\alpha+\alpha'+\beta')^2 - 4(\alpha\beta+\alpha'\beta')\right).$$

We retrieve the same separation as in the large ε regime. Denoting by $\beta_c = \frac{(\alpha+\alpha'+\beta')^2-4\alpha'\beta'}{4\alpha}$, we have at order one in ε:

$$\sigma_\pm^N \underset{\varepsilon\to 0}{\approx} \frac{\varepsilon}{2\rho_N} \times \begin{cases} -\alpha'-\beta'-\alpha \pm \sqrt{(\alpha'+\beta'+\alpha)^2 - 4(\alpha\beta+\alpha'\beta')} & \text{if} \quad \beta<\beta_c \\ -\alpha'-\beta'-\alpha \pm i\sqrt{4(\alpha\beta+\alpha'\beta') - (\alpha'+\beta'+\alpha)^2} & \text{if} \quad \beta>\beta_c \\ -\alpha'-\beta'-\alpha & \text{if} \quad \beta=\beta_c \end{cases}. \tag{B.17}$$

In the limit $\varepsilon \to 0$, $\rho_N = z_{\max}$ and we retrieve the equations given in the text. Figure 12 shows the adequacy between the theoretical estimate and the actual largest eigenvalue (obtained through numerical simulations of the matrix $\mathbb{D}$) as $\varepsilon \to 0$.

Appendix C: Blocks of the stability matrix

In this section, we give the values of the perturbation terms for the blocks of the stability matrix. We introduce several notations for quantities that simplify in the case of an undirected network with homogeneous productivity factors. Finally, we use the bra (resp. ket) notation to refer to a row (resp. column) vector $|v\rangle$ and we denote by v_i its i^{th} component.

1. Perturbation of $\mathbf{p}_{\mathbf{eq}}$ and $\boldsymbol{\gamma}_{\mathbf{eq}}$

a. Prices

Equilibrium prices are easily obtained by applying $\boldsymbol{\mathcal{M}}^{-1}$ to the vector $|V\rangle$ yielding

$$\begin{aligned} p_{eq,j} = &\frac{1}{\varepsilon}\langle \ell_N \mid V\rangle\, r_{N,j} + \sum_{\nu=1}^{N-1}\frac{\langle l_\nu \mid V\rangle}{\rho_N-\rho_\nu} r_{\nu,j} - \varepsilon\sum_{\nu=1}^{N-1}\frac{\langle l_\nu \mid V\rangle}{(\rho_N-\rho_\nu)^2} r_{\nu,j} + \varepsilon^2\sum_{\nu=1}^{N-1}\frac{\langle l_\nu \mid V\rangle}{(\rho_N-\rho_\nu)^3} r_{\nu,j} \\ &- \varepsilon^3\sum_{\nu=1}^{N-1}\frac{\langle l_\nu \mid V\rangle}{(\rho_N-\rho_\nu)^4} r_{\nu,j} \\ := &\frac{1}{\varepsilon}\pi_{-1}^l(V)_j + \pi_0^l(V)_j - \varepsilon\pi_1^l(V)_j + \varepsilon^2\pi_2^l(V)_j - \varepsilon^3\pi_3^l(V)_j, \end{aligned} \tag{1.a}$$

where we introduced for $i \geq 0$

$$\pi^l_{-1}(V) = \langle \ell_N \mid V \rangle \, |r_N\rangle \ , \ \pi^l_i(V) = \sum_{\nu=1}^{N-1} \frac{\langle l_\nu \mid V \rangle}{(\rho_N - \rho_\nu)^{i+1}} \, |r_\nu\rangle \, .$$

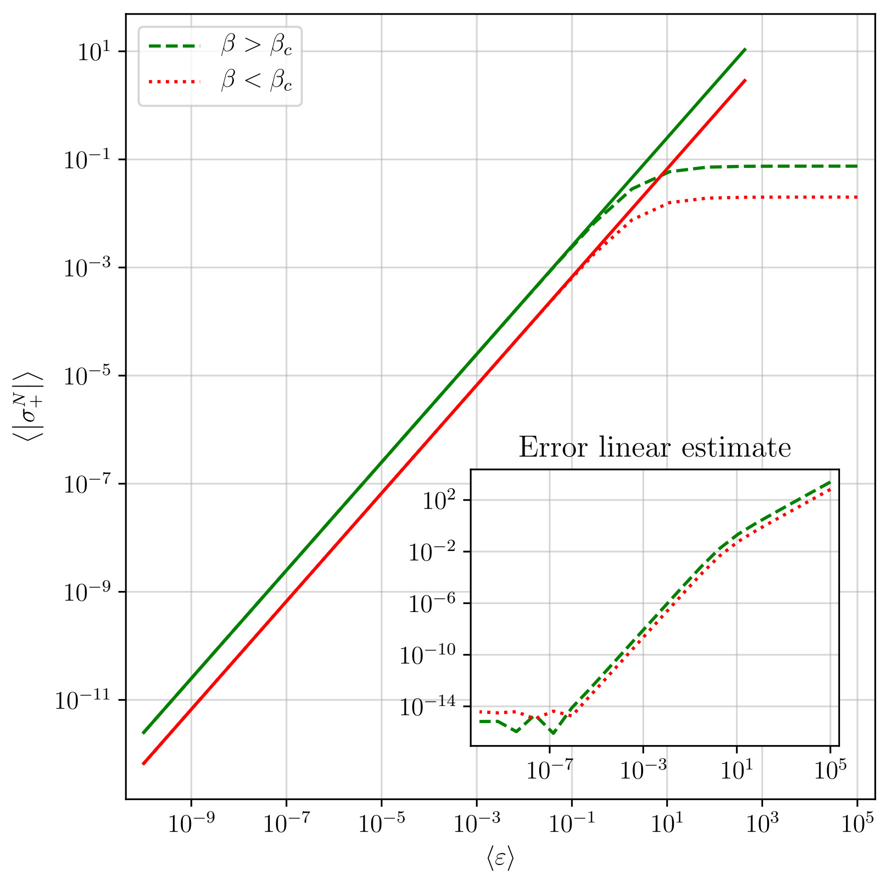

FIG. 12: Plain line: simulated smallest eigenvalue. Dashed: theoretical estimate for $\varepsilon \ll 1$. Error plot give the error between the linear estimate from (B.17) and the simulated eigenvalue. Simulations are made for an economy on $N = 100$ firms on a 3-regular undirected network with unit weights. We generate 50 such economies and average out the eigenvalue of $\mathbb{D}$ closest to 0 in real part.

b. Productions

Equilibrium productions can be a little trickier to obtain. We first derive three useful identities to simplify calculations. For $s = 1, \dots, n$, we have

$$\frac{1}{\varepsilon p_{eq,s}} = \frac{1}{\pi^l_{-1}(V)_s} - \varepsilon \frac{\pi^l_0(V)_s}{\left(\pi^l_{-1}(V)_s\right)^2} + \frac{\varepsilon^2}{\pi^l_{-1}(V)_s}\left(\frac{\pi^l_1(V)_s}{\pi^l_{-1}(V)_s} + \left(\frac{\pi^l_0(V)_s}{\pi^l_{-1}(V)_s}\right)^2\right) - \frac{\varepsilon^3}{\pi^l_{-1}(V)_s}\left(\frac{\pi^l_2(V)_s}{\pi^l_{-1}(V)_s} + 2\frac{\pi^l_0(V)_s\pi^l_1(V)_s}{\pi^l_{-1}(V)_s} + \left(\frac{\pi^l_0(V)_s}{\pi^l_{-1}(V)_s}\right)^3\right) \quad \text{(i)}$$

$$\frac{1}{p_{eq,s}} = \frac{\varepsilon}{\pi^l_{-1}(V)_s} - \varepsilon^2 \frac{\pi^l_0(V)_s}{\left(\pi^l_{-1}(V)_s\right)^2} + \frac{\varepsilon^3}{\pi^l_{-1}(V)_s}\left(\frac{\pi^l_1(V)_s}{\pi^l_{-1}(V)_s} + \left(\frac{\pi^l_0(V)_s}{\pi^l_{-1}(V)_s}\right)^2\right) \quad \text{(ii)}$$

$$\frac{\varepsilon}{p_{eq,s}} = \frac{\varepsilon^2}{\pi^l_{-1}(V)_s} - \varepsilon^3 \frac{\pi^l_0(V)_s}{\left(\pi^l_{-1}(V)_s\right)^2} \quad \text{(iii)}$$

$$\frac{\varepsilon^2}{peq_s} = \frac{\varepsilon^3}{\pi^l_{-1}(V)_s} \quad \text{(iv)}$$

$$\frac{\varepsilon^3}{peq_s} = o(\varepsilon^3). \quad \text{(v)}$$

These results allow to write the equilibrium productions with $\psi_{eq,i} = \frac{\mu\theta_i}{p_{\text{eq},i}}$

$$\begin{aligned}
\gamma_{eq,j} &= \frac{\mu}{\varepsilon}\left\langle r_N \mid \psi_{eq}\right\rangle l_{N,j} + \mu\sum_{\nu=1}^{N-1}\frac{\left\langle r_\nu \mid \psi_{eq}\right\rangle}{\rho_N-\rho_\nu} l_{\nu,j} - \varepsilon\mu\sum_{\nu=1}^{N-1}\frac{\left\langle r_\nu \mid \psi_{eq}\right\rangle}{(\rho_N-\rho_\nu)^2} l_{\nu,j} + \varepsilon^2\mu\sum_{\nu=1}^{N-1}\frac{\left\langle r_\nu \mid \psi_{eq}\right\rangle}{(\rho_N-\rho_\nu)^3} l_{\nu,j} \\
&\quad - \varepsilon^3\mu\sum_{\nu=1}^{N-1}\frac{\left\langle r_\nu \mid \psi_{eq}\right\rangle}{(\rho_N-\rho_\nu)^4} l_{\nu,j} \\
&= \mu l_{N,j}\sum_{s=1}^{n}\frac{r_{n,s}\theta_s}{\pi^l_{-1}(V)_s} + \varepsilon\mu\sum_{s=1}^{n}\left\{\sum_{\nu=1}^{N-1}\frac{l_{\nu,j}r_{\nu,s}\theta_s}{(\rho_N-\rho_\nu)\pi^l_{-1}(V)_s} - \frac{l_{N,j}r_{n,s}\theta_s\pi^l_0(V)_s}{\pi^l_{-1}(V)_s^2}\right\} \\
&\quad + \varepsilon^2\mu\sum_{s=1}^{n}\left\{\frac{\pi^l_1(V)_s}{\pi^l_{-1}(V)_s} + \left(\frac{\pi^l_0(V)_s}{\pi^l_{-1}(V)_s}\right)^2 - \sum_{\nu=1}^{N-1}\frac{l_{\nu,j}r_{\nu,s}\theta_s}{(\rho_N-\rho_\nu)\pi^l_{-1}(V)_s}\left(\frac{1}{\rho_N-\rho_\nu} + \frac{\pi^l_0(V)_s}{\pi^l_{-1}(V)_s}\right)\right\} \\
&\quad + \varepsilon^3\mu\sum_{s=1}^{n}\left\{\sum_{\nu=1}^{N-1}\frac{l_{\nu,j}r_{\nu,s}\theta_s}{(\rho_N-\rho_\nu)\pi^l_{-1}(V)_s}\left(\frac{\pi^l_1(V)_s}{\pi^l_{-1}(V)_s} + \left(\frac{\pi^l_0(V)_s}{\pi^l_{-1}(V)_s}\right)^2 + \frac{\pi^l_0(V)_s}{(\rho_N-\rho_\nu)\pi^l_{-1}(V)_s}\right.\right. \\
&\qquad\qquad \left.\left. + \frac{1}{(\rho_N-\rho_\nu)^3\pi^l_{-1}(V)_s}\right)\right\} \\
&:= \mu f_{0,j} + \varepsilon\mu f_{1,j} + \varepsilon^2\mu f_{2,j} + \varepsilon^3\mu f_{3,j}.
\end{aligned} \quad \text{(1.b)}$$

2. Stability blocks

The next step is to perturb the stability matrix itself. This yields no particular difficulty but calculations are a bit long so that we only give the results for the different blocks. We denote by τg_k the coefficients of the expansion of $z\gamma$ where $\tau_i = \rho_N e_{N,i}/\left\langle e_N \mid V\right\rangle$ for an undirected network. We have

$$
\begin{aligned}
(D_1^{(0)})_{ij} &= -\alpha' \frac{1}{\rho_N} M_{ij} \\
(D_1^{(1)})_{ij} &= -\alpha\mu \frac{\theta_i}{\tau_i \pi_{-1}^l(V)_i} \delta_{ij} + \frac{\alpha'}{\rho_N^2} M_{ij} - \frac{\alpha'}{\rho_N} \delta_{ij} \\
(D_1^{(2)})_{ij} &= +\alpha\mu \frac{\theta_i}{\tau_i \pi_{-1}^l(V)_i} \left(g_{1,i} + \frac{\pi_0^l(V)_i}{\pi_{-1}^l(V)_i} \right) \delta_{ij} - \frac{\alpha'}{\rho_N^3} M_{ij} + \frac{\alpha'}{\rho_N^2} \delta_{ij} \\
(D_1^{(3)})_{ij} &= -\alpha\mu\varepsilon \frac{\theta_i}{\tau_i \pi_{-1}^l(V)_i} \left(g_{1,i}^2 - g_{2,i} + \frac{\pi_0^l(V)_i}{\pi_{-1}^l(V)_i} g_{1,i} + \frac{\pi_1^l(V)_i}{\pi_{-1}^l(V)_i} + \left(\frac{\pi_0^l(V)_i}{\pi_{-1}^l(V)_i} \right)^2 \right) \delta_{ij} + \frac{\alpha'}{\rho_N^4} M_{ij} - \frac{\alpha'}{\rho_N^3} \delta_{ij},
\end{aligned}
$$

$$
\begin{aligned}
(D_2^{(-1)})_{ij} &= -\alpha \frac{\pi_{-1}^l(V)_i}{\tau_i} M_{ji} \\
(D_2^{(0)})_{ij} &= \frac{\alpha}{\tau_i} \left[M_{ji} \left(\pi_{-1}^l(V)_i g_{1,i} - \pi_0^l(V)_i \right) - \pi_{-1}^l(V)_i \delta_{ij} \right] \\
(D_2^{(1)})_{ij} &= \frac{\alpha}{\tau_i} \left[M_{ji} \left(\pi_{-1}^l(V)_i g_{1,i}^2 + 2\pi_{-1}^l(V)_i g_{1,i} g_{2,i} + \pi_0^l(V)_i g_{1,i} - \pi_{-1}^l(V)_i g_{2,i} - \pi_1^l(V)_i \right) \right. \\
&\quad \left. + \left(\pi_{-1}^l(V)_i g_{1,i} - \pi_0^l(V)_i \right) \delta_{ij} \right] \\
(D_2^{(2)})_{ij} &= \frac{\alpha}{\tau_i} \left[M_{ji} \left(\pi_1^l(V)_i g_{1,i} + \pi_2^l(V)_i - \pi_{-1}^l(V)_i \left(g_{3,i} + g_{1,i}^3 \right) + \pi_0^l(V)_i \left(g_{1,i}^2 - g_{2,i} \right) \right) \right. \\
&\quad \left. + \left(\pi_{-1}^l(V)_i \left(g_{1,i}^2 - g_{2,i} \right) - \pi_0^l(V)_i g_{1,i} - \pi_1^l(V)_i \right) \delta_{ij} \right],
\end{aligned}
$$

$$
\begin{aligned}
(D_3^{(1)})_{ij} &= \frac{\mu\beta f_{0,i}}{\rho_N \pi_{-1}^l(V)_i} M_{ij} \\
(D_3^{(2)})_{ij} &= \frac{\mu\beta f_{0,i}}{\rho_N} \left[\left(\frac{g_{1,i}}{\pi_{-1}^l(V)_i} - \frac{\pi_0^l(V)_i}{(\pi_{-1}^l(V)_i)^2} + \frac{g_{1,i}}{\rho_N} - \frac{\pi_0^l(V)_i}{\rho_N \pi_{-1}^l(V)_i} \right) M_{ij} + \frac{1}{\pi_{-1}^l(V)_i} \delta_{ij} \right] - \beta' \frac{\mu\theta_i}{\rho_N \pi_{-1}^l(V)_i^2} \delta_{ij} \\
(D_3^{(3)})_{ij} &= \frac{\mu\beta f_{0,i}}{\rho_N} \left[M_{ij} \left(\frac{\pi_1^l(V)_i}{(\pi_{-1}^l(V)_i)^2} + \frac{(\pi_0^l(V)_i)^2}{(\pi_{-1}^l(V)_i)^3} - \frac{\pi_0^l(V)_i}{(\pi_{-1}^l(V)_i)^2} g_{1,i} + \frac{g_{1,i}}{\rho_N \pi_{-1}^l(V)_i} - \frac{\pi_0^l(V)_i}{\rho_N (\pi_{-1}^l(V)_i)^2} \right) \right. \\
&\quad \left. + \left(\frac{g_{1,i}}{\pi_{-1}^l(V)_i} - \frac{\pi_0^l(V)_i}{(\pi_{-1}^l(V)_i)^2} + \frac{1}{\rho_N} \left(g_{1,i} - \frac{\pi_0^l(V)_i}{\pi_{-1}^l(V)_i} \right) \right) \delta_{ij} \right] + \beta' \frac{\mu\theta_i}{\rho_N \pi_{-1}^l(V)_i^2} \left(\frac{2\pi_0^l(V)_i}{\pi_{-1}^l(V)_i} + \frac{1}{\rho_N} \right) \delta_{ij},
\end{aligned}
$$

$$
\begin{aligned}
(D_4^{(0)})_{ij} &= -\beta' \frac{1}{\rho_N} M_{ij} \\
(D_4^{(1)})_{ij} &= \frac{\beta'}{\rho_N^2} M_{ij} - \frac{\beta'}{\rho_N} \delta_{ij} \\
(D_4^{(2)})_{ij} &= -\frac{\beta'}{\rho_N^3} M_{ij} + \frac{\beta'}{\rho_N^2} \delta_{ij} \\
(D_4^{(3)})_{ij} &= \frac{\beta'}{\rho_N^4} M_{ij} - \frac{\beta'}{\rho_N^3} \delta_{ij}.
\end{aligned}
$$

Appendix D: Critical volatility of prices and outputs with fluctuations

1. General computation for marginally stable linear stochastic systems

In this section, we consider a general evolution of a vector $\mathbf{U}(t)$ given by the linear stochastic equation

$$\frac{\mathrm{d}\mathbf{U}(t)}{\mathrm{d}t} = \mathbb{D}\mathbf{U}(t) + \boldsymbol{\xi}(t), \tag{D.1}$$

where $\mathbb{D}$ is a real $N \times N$ matrix and $\xi(t)$ is a Gaussian correlated noise such that

$$\langle \xi_i(t) \rangle = 0 \tag{D.2}$$

$$\langle \xi_i(t)\xi_j(s) \rangle = 2\sigma^2 \delta_{ij} G\left(|t-s|\right). \tag{D.3}$$

We assume the dynamical matrix $\mathbb{D}$ to be diagonalizable with real eigenvalues[20] such that

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1} < \lambda_N := -\varepsilon < 0.$$

Negative eigenvalues means that the system is stable i.e $\langle \|\mathbf{U}(t)\| \rangle \to 0$ as $t \to \infty$ for any initial condition. We assume that $\varepsilon \to 0$ and show that the volatility of $\mathbf{U}(t)$ increases as $\varepsilon^{-1/2}$. We introduce the eigenvectors $\mathbf{e}_\nu$ associated to λ_ν and we express $\mathbf{U}$ into the diagonal basis

$$\mathbf{U}(t) = \sum_{\nu=1}^{N} u_\nu \mathbf{e}_\nu. \tag{D.4}$$

Injecting this expression into (D.1), we get and evolution equation for the components of $\mathbf{U}(t)$ in the diagonal basis

$$\frac{d}{dt} u_\nu = \lambda_\nu u_\nu + \boldsymbol{\xi}(t) \cdot \mathbf{e}_\nu. \tag{D.5}$$

We can give an explicit solution for these components

$$u_\nu(t) = e^{\lambda_\nu t} \left[u_\nu(0) + \int_0^t ds\, e^{-\lambda_\nu s} \boldsymbol{\xi}(t) \cdot \mathbf{e}_\nu \right], \tag{D.6}$$

and focus on u_N since this is the component which yields the $\varepsilon^{-1/2}$-volatility. To do so, we compute the average value of $u_N(t)^2 - \langle u_N(t) \rangle^2$

$$\begin{aligned}
\left\langle u_N(t)^2 - \langle u_N(t) \rangle^2 \right\rangle &= e^{-2\varepsilon t} \left\langle \left[u_N(0) + \int_0^t ds\, e^{\varepsilon s} \boldsymbol{\xi}(s) \cdot \mathbf{e}_N \right]^2 \right\rangle - u_N(0)^2 e^{-\varepsilon t} \\
&= e^{-2\varepsilon t} \left[u_N(0)^2 + 2u_N(0) \int_0^t ds\, e^{\varepsilon s} \langle \boldsymbol{\xi}(s) \cdot \mathbf{e}_N \rangle + \int ds ds'\, e^{\varepsilon(s+s')} \langle (\boldsymbol{\xi}(s) \cdot \mathbf{e}_N)(\boldsymbol{\xi}(s') \cdot \mathbf{e}_N) \rangle \right] - u_N(0)^2 e^{-\varepsilon t} \\
&= e^{-\varepsilon t} \sum_{j,k} e_{N,j} e_{N,k} \int ds ds'\, e^{\varepsilon(s+s')} \langle \xi_j(s) \xi_j(s') \rangle \\
&= 2\sigma^2 \|\mathbf{e}_N\|^2 e^{-\varepsilon t} \int ds ds'\, e^{\varepsilon(s+s')} G(|s'-s|),
\end{aligned}$$

we substitute $\tau = s' - s$ in the s integral to get

$$= 2\sigma^2 \|\mathbf{e}_N\|^2 e^{-\varepsilon t} \int_0^t ds'\, e^{2\varepsilon s'} \int_0^{s'-t} d\tau e^{-\varepsilon \tau} G(\tau).$$

Using the quick decay of the exponential term in the τ integral, we can expend the integration domain...

$$\approx 2\sigma^2 \|\mathbf{e}_N\|^2 e^{-\varepsilon t} \int_0^t ds'\, e^{2\varepsilon s'} \int_0^{\infty} d\tau e^{-\varepsilon \tau} G(\tau),$$

... and perform the integration over s' with an approximately vanishing exponential remainder

$$\approx \frac{\sigma^2 \|\mathbf{e}_N\|^2}{\varepsilon} \int_0^{\infty} d\tau e^{-\varepsilon \tau} G(\tau).$$

Denoting be τ_ξ the typical correlation time of G, we see that

---

[20] The case with complex eigenvalues leads to the same conclusions. One must only take into account the fact that, since $\mathbb{D}$ is real, eigenvalues and eigenvectors will be conjugated so that their are two eigenvalues that are smallest in real parts. We make the same ordering of eigenvalues replacing the λ's by their real parts.

- if $\varepsilon\tau_\xi \ll 1$ (meaning that G correlates on short time-scales) then $G(\tau) \sim \delta(0)$ such that

$$\int_0^\infty d\tau e^{-\varepsilon\tau} G(\tau) \approx 1,$$

- if $\varepsilon\tau_\xi \gg 1$ (meaning that G correlates on long time-scales) then $G(\tau) \sim G(0)$ on the decay time of the exponential such that

$$\int_0^\infty d\tau e^{-\varepsilon\tau} G(\tau) \approx \frac{G(0)}{\varepsilon}.$$

Finally, the volatility of $\mathbf{U}(t)$ behaves as

$$\sqrt{\left\langle u_N(t)^2 - \langle u_N(t)\rangle^2 \right\rangle} \propto \begin{cases} \varepsilon^{-1/2} & \text{if} \quad \varepsilon\tau_\xi \ll 1 \\ \varepsilon^{-1} & \text{if} \quad \varepsilon\tau_\xi \gg 1 \end{cases}. \tag{D.7}$$

Note also that this result generalizes to discrete time processes (which is of interest in the case of the general ABM that we present)

$$\mathbf{U}_{t+1} = \mathbb{D}\mathbf{U}_t + \boldsymbol{\xi}_t. \tag{D.8}$$

The marginal stability condition can be written as $\lambda_N = 1 - \varepsilon$ [21] with $\varepsilon \to 0$. We can carry out the same kind of computation and derive the same result depending on the behavior of the quantity $\sum_{\tau \geq 0} (1-\varepsilon)^\tau G(\tau)$.

2. Computation of the volatility induced by gaussian shocks on productivity factors

If we consider shocks on productivity factors $z_i(t) = z_i + \xi_i(t)$ with $\xi(t)$ given as before, we can linearize the dynamics of the naive model in both small deviations from equilibrium and small shocks. The stochastic equation that we retrieve reads

$$\frac{\mathrm{d}\mathbf{U}(t)}{\mathrm{d}t} = \mathbb{D}\mathbf{U}(t) + \boldsymbol{\Xi}(t), \tag{D.9}$$

with a noise $\boldsymbol{\Xi}$ of the form

$$\boldsymbol{\Xi}(t) = \begin{pmatrix} -\frac{\alpha+\alpha'}{z_i} \mathbf{p}_{\mathrm{eq}} \circ \boldsymbol{\xi}(t) \\ \frac{\beta-\beta'}{z_i} \boldsymbol{\gamma}_{\mathrm{eq}} \circ \boldsymbol{\xi}(t) \end{pmatrix} \underset{\varepsilon\to 0}{\sim} \begin{pmatrix} -\frac{\alpha+\alpha'}{\varepsilon\rho_N} (\boldsymbol{\ell}_N \cdot \mathbf{V}) \mathbf{r}_N \circ \boldsymbol{\xi}(t) \\ \frac{\beta-\beta'}{(\boldsymbol{\ell}_N\cdot\mathbf{V})\rho_N} \mathbf{l}_N \circ \boldsymbol{\xi}(t) \end{pmatrix}, \tag{D.10}$$

with notations from C. The correlations of this noise are slightly more complicated than before

$$\langle \Xi_i(t) \Xi_j(s) \rangle = \sigma^2 G(|t-s|) \times \begin{cases} \delta_{ij} \left(\frac{(\alpha+\alpha')(\boldsymbol{\ell}_N\cdot\mathbf{V})}{\rho_N}\right)^2 r_{N,i} r_{N,j} \varepsilon^{-2} & \text{if} \quad i,j \leq n \\ \delta_{ij} \left(\frac{\beta-\beta'}{(\boldsymbol{\ell}_N\cdot\mathbf{V})\rho_N}\right)^2 l_{N,i} l_{N,j} & \text{if} \quad i,j > n \\ -\delta_{i,j-n} \frac{(\beta-\beta')(\alpha+\alpha')}{\rho_N^2} r_{N,i} l_{N,j} \varepsilon^{-1} & \text{if} \quad i \leq n,\ j > n \\ -\delta_{i-n,j} \frac{(\beta-\beta')(\alpha+\alpha')}{\rho_N^2} l_{N,i} r_{N,j} \varepsilon^{-1} & \text{if} \quad i > n,\ j \leq n \end{cases}. \tag{D.11}$$

The dynamical matrix of the naive model with an undirected network two eigenvalues yields $\sigma_N^\pm = k^\pm \varepsilon \to 0$ with associated eigenvectors $\boldsymbol{\Sigma}_N^\pm = (\mathbf{e}_N, \boldsymbol{\nu}^\pm \varepsilon)^t$ for undirected networks. We assume $\beta < \beta_c$ so that the marginal eigenvalues are real as well as their eigenvectors. It follows that, at leading order in ε, the volatility of the marginal components of $\mathbf{U}(t)$ behaves as $\varepsilon^{-3/2}$. Indeed

$$\left\langle u_N^\pm(t)^2 - \left\langle u_N^\pm(t)\right\rangle^2 \right\rangle = \frac{\sigma^2}{\left(k^\pm\right)^2 \varepsilon^3} \left(\frac{(\alpha+\alpha')(\mathbf{e}_N \cdot \mathbf{V})}{\rho_N}\right)^2 \mathscr{H}(\mathbf{e}_N) \int_0^\infty d\tau e^{-\varepsilon\tau} G\left(\frac{\tau}{\nu^\pm}\right),$$

---

[21] Or more generally for complex eigenvalues $\lambda_N = r_N e^{i\theta_N}$ with $r_N = 1 - \varepsilon$.

where $\mathscr{H}$ represents the inverse participation ratio. To retrieve the volatility as $\varepsilon^{-1/2}$ we may rescale $\delta p_i(t)$ (resp. $\delta\gamma_i(t)$) by $p_{eq,i}$ (resp. $\gamma_{eq,i}$). Denoting by $\mathbf{w}_i$ the i^{th} canonical vector of $\mathbb{R}^{2N}$, we have

$$\begin{aligned}
\mathrm{Var}\left(\frac{\delta p_i(t)}{p_{eq,i}}\right) &= p_{eq,i}^{-2}\mathrm{Var}\left(\sum_{\substack{k=1\\ \tau=\pm}}^{N} u_k^\tau(t)(\mathbf{\Sigma}_k^\pm \cdot \mathbf{w}_i)\right) \\
&\underset{\substack{\varepsilon\to 0\\ \varepsilon t\ll 1}}{\approx} \frac{\varepsilon^2}{e_{N,i}^2(\mathbf{e}_N\cdot\mathbf{V})^2}\mathrm{Var}\left(u_N^+(t)e_{N,i} + u_N^-(t)e_{N,i}\right) \\
&= \frac{\varepsilon^2}{(\mathbf{e}_N\cdot\mathbf{V})^2}\left[\mathrm{Var}\left(u_N^+(t)\right) + \mathrm{Var}\left(u_N^-(t)\right) + 2\mathrm{Cov}\left(u_N^+(t), u_N^-(t)\right)\right] \\
&\propto \frac{1}{\varepsilon}; \\
\mathrm{Var}\left(\frac{\delta \gamma_i(t)}{\gamma_{eq,i}}\right) &= \gamma_{eq,i}^{-2}\mathrm{Var}\left(\sum_{\substack{k=1\\ \tau=\pm}}^{N} u_k^\tau(t)(\mathbf{\Sigma}_k^\pm \cdot \mathbf{w}_{i+N})\right) \\
&\underset{\substack{\varepsilon\to 0\\ \varepsilon t\ll 1}}{\approx} \frac{(\mathbf{e}_N\cdot\mathbf{V})^2}{e_{N,i}^2}\mathrm{Var}\left(u_N^+(t)\nu^+\varepsilon e_{N,i} + u_N^-(t)\nu^-\varepsilon e_{N,i}\right) \\
&= (\mathbf{e}_N\cdot\mathbf{V})^2\varepsilon^2\left[(\nu^+)^2\mathrm{Var}\left(u_N^+(t)\right) + (\nu^-)^2\mathrm{Var}\left(u_N^-(t)\right) + 2\nu^+\nu^-\mathrm{Cov}\left(u_N^+(t), u_N^-(t)\right)\right] \\
&\propto \frac{1}{\varepsilon}.
\end{aligned}$$

Appendix E: Code for the simulation

1. Objects

The simulation uses an object-oriented approach. Each object has attributes (parameters of the model, or other quantities that we can infer from the parameters) along with methods that carry out more complicated tasks. There are four objects in our simulation. The first two are the `firms` and `household` classes, and correspond to the smallest entities in our model (the agents). The `economy` class carries all of the static information – essentially the different parameters describing the interactions between the agents – of the model, along with instances of the `firms` and `household` classes. Finally, the `dynamics` class handles the evolution of the model by storing the time series of prices, productions and so on, along with the different methods that allow the model to move forward in time; the `dynamics` class contains an instance of the `economy` class that is used for the simulation. In our framework, classes need one another to function properly. For instance, firms need to know the input-output network to compute their optimal quantities as in II.3. As a consequence, some methods take instances of classes in their arguments, as is the case in e.g. the `compute_optimal_quantities` method of the firm `firms` class. We detail this last method as an example in Procedure 1.

Procedure 1 The `firms` class

class FIRMS

attributes

z: 1d-ARRAY ▷ Productivity factors
α: 1d-ARRAY ▷ Log elasticity factors of prices to surplus
α': 1d-ARRAY ▷ Log elasticity factors of prices to profits
β: 1d-ARRAY ▷ Log elasticity factors of productions to profits
β': 1d-ARRAY ▷ Log elasticity factors of productions to surplus
ω: FLOAT ▷ Log elasticity factors of wages to labor-market tensions

methods

1d-ARRAY `update_prices`$(\mathbf{p}(t), \mathscr{S}(t), \mathscr{D}(t), \mathscr{G}(t), \mathscr{L}(t))$ ▷ Update prices according to (IV.9)
FLOAT `update_wages`$(L^s(t), L^d(t))$ ▷ Update wages according to (IV.10)
1d-ARRAY `compute_targets`$(\mathbf{p}(t), \mathbb{E}_t[\mathbf{Q}(t)], \mathscr{S}(t), \boldsymbol{\gamma}(t))$ ▷ Compute targets according to (IV.1)
4d-ARRAY `compute_forecasts`$(\mathbf{p}(t), \mathbb{E}_t[\mathbf{Q}(t)], \mathscr{S}(t))$ ▷ Compute forecasts according to (IV.18, IV.17)
1d-ARRAY `compute_optimal_quantities`$(\hat{\boldsymbol{\gamma}}(t), \mathbf{p}(t),$ `economy`$)$ ▷ Compute optimal quantities according to (II.3)
4d-ARRAY `compute_profits_balance`$(\mathbf{p}(t), \mathbf{Q}^e(t), \mathscr{S}(t), \mathscr{D}(t))$ ▷ Compute profits and balance according to (IV.7, IV.8)

end class

2. Pseudo-code to execute one step of the time-line

In Procedure 2, we present a pseudo-code to execute one time-step of the model. In order to get the full dynamics, one loops over this procedure during a time T, after a careful initialization. To initialize, one needs to give the dynamics an initial value for prices $p_i(t=1)$, wages $p_0(t=1)$, production levels $\gamma_i(t=1)$, targets $\hat{\gamma}_i(t=2)$, stocks $I_{ij}(t=1)$ and savings $S(t=1)$. One also needs to carry out the initial planning by the household to have a value for $\mathbb{E}_1[C^d(t=1)]$ and $L^s(t=1)$. This entire process of initialization and loop over Procedure 2 in encapsulated into a class `dynamics`. This class stores the entire history of the most fundamental quantities (prices, demand matrix...) into array of the appropriate size, and uses reconstruction methods for all the inferable quantities (productions, targets, profits...). This way, the algorithm is quicker and less memory-demanding. Finally, Procedure 2 is quite detailed compared to the actual implementation. Bearing in mind complexity issues, most of the loops of Procedure 2 are implemented in a single line through matrix multiplication. Using the result $(\Delta M)_{ij} = \Delta_{ii} M_{ij}$ and $(M\Delta)_{ij} = \Delta_{jj} M_{ij}$ with Δ a diagonal matrix, one can implement the procedure to go from demanded quantities to exchanged quantities as

$$
\begin{aligned}
\mathbf{Q}(t) =& \boldsymbol{\Delta}\left(\left[\min\left(1, \frac{B(t)}{\sum_i p_i(t) C_i^d(t) \min\left(1, \mathscr{S}_i(t)/\mathscr{D}_i(t)\right)}\right), 1, \dots, 1\right]\right) \\
& \mathbf{Q}^d(t) \\
& \boldsymbol{\Delta}\left(\left[\min\left(1, \frac{L^s(t)}{L^d(t)}\right), \min\left(1, \frac{\mathscr{S}_1(t)}{\mathscr{D}_1(t)}\right), \dots, \min\left(1, \frac{\mathscr{S}_N(t)}{\mathscr{D}_N(t)}\right)\right]\right)
\end{aligned}
$$

where we use the convention $Q^d_{00} = Q^e_{00} = 0$. Finally, we denote by ∂_i^{in} (resp. ∂_i^{out}) the set of suppliers (resp. buyers) of firm i.

Procedure 2 Fundamental time-step

Phase 1 - Planning

Input: $L^s(t)$, $\boldsymbol{\gamma}(t)$, $\mathbf{p}(t)$, $\mathbf{I}(t)$, $\mathbf{Q}(t)$

for all firms i **do**

$\mathscr{S}_i(t) \leftarrow z_i\gamma_i(t) + I_{ii}(t)$

$\hat{\gamma}_i(t+1) \leftarrow \texttt{compute\_targets}(\mathbf{p}(t), \mathbb{E}_t[\mathbf{Q}(t)], \mathscr{S}_i(t), \gamma_i(t))$ ▷ Computation of targets according to forecasts

$\widehat{\boldsymbol{Q}}(t) \leftarrow \texttt{compute\_optimal\_quantities}(\hat{\boldsymbol{\gamma}}(t+1), \mathbf{p}(t), \texttt{economy})$

for all firms i **do**

$Q^d_{i0}(t) := \ell^d_i(t) \leftarrow \widehat{Q}_{i0}(t)$

for all firms $j \in \partial^{in}_i$ **do**

$Q^d_{ij}(t) \leftarrow \max\left(0, \widehat{Q}_{ij} - I_{ij}\right)$

Output: $\mathscr{S}_i(t)$, $\hat{\boldsymbol{\gamma}}_i(t)$, $\widehat{\mathbf{Q}}(t)$, $\mathbf{Q}^d(t)$, $\ell^d(t)$, $S(t)$

Phase 2 - Exchanges & Updates

Input: $\mathscr{S}_i(t)$, $\widehat{\mathbf{Q}}(t)$, $\mathbf{Q}^d(t)$, $\ell^d(t)$, $C^d_i(t)$, $\mathbb{E}_t[B(t)]$

for all firms i **do**

$Q_{i0} := \ell_i \leftarrow \ell^d_i \min\left(1, \frac{L^s(t)}{L^d(t)}\right)$ ▷ Workers are hired

$B(t) \leftarrow S(t) + \sum_{i=1}^n \ell^e_i(t)$ ▷ Wages are paid

for all firms i **do**

$Q^d_{0i} := C^d_i(t) \leftarrow C^d_i(t)\left(\nu + (1-\nu)\min\left(1, \frac{B(t)}{\mathbb{E}_t[B(t)]}\right)\right)$ ▷ Household adjusts its consumption demands ($\nu = 1$ in this paper)

$\mathscr{D}_i(t) \leftarrow \sum_{j\in\partial^{out}_i} Q^d_{ji}(t)$ ▷ Firms compute their total demand

for all firms $j \in \partial^{out}_i$ **do**

$Q_{ji} \leftarrow Q^d_{ji}\min\left(1, \frac{\mathscr{S}_i(t)}{\mathscr{D}_i(t)}\right)$ ▷ Exchanges of goods are carried out

$C^r_i(t) \leftarrow C^d_i \min\left(1, \frac{\mathscr{S}_i(t)}{\mathscr{D}_i(t)}\right)\min\left(1, \frac{B(t)}{\mathbf{p}(t)\cdot\mathbf{C}^e(t)}\right)$ ▷ Household consumes according to its budget

$\mathscr{G}_i(t)$, $\mathscr{L}_i(t) \leftarrow p_i(t)\sum_{j\in\partial^{out}_i} Q^e_{ji}(t)$, $\sum_{j\in\partial^{in}_i} Q^e_{ij}(t)p_j(t)$

$S(t+1) \leftarrow B(t) - \mathbf{p}(t)\cdot\mathbf{C}^e(t)$ ▷ The household saves unspent money

for all firms i **do**

$Q^a_{i0}(t) \leftarrow Q_{i0}(t)$ ▷ Labor available for production is the hired workforce

for all firms $j \in \partial^{in}_i$ **do**

$Q^a_{ij}(t) \leftarrow Q_{ij}(t) + \min\left(\widehat{Q}_{ij}(t), I_{ij}(t)\right)$ ▷ Available goods depend on exchanges and current stocks

$p_0(t+1) \leftarrow \texttt{update\_wage}(L^s(t), L^d(t), \omega)$ ▷ Wage is updated

for all firms i **do**

$p_i(t+1) \leftarrow \texttt{update\_price}(\mathscr{S}_i(t), \mathscr{D}_i(t), \mathscr{G}_i(t), \mathscr{L}_i(t), \alpha, \alpha', \beta, \beta')$ ▷ Prices are updated

Output: $\mathbf{Q}^e(t)$, $\mathbf{Q}^p(t)$, $S(t+1)$, $B(t)$, $p_0(t+1)$, $\mathscr{G}_i(t)$,$\mathscr{G}_i(t)$

Phase 3 - Production

Input: $S(t+1)$, $B(t)$, $p_0(t+1)$, $\mathscr{G}_i(t)$,$\mathscr{G}_i(t)$,$\mathbf{Q}^e(t)$, $\mathbf{Q}^p(t)$, $\widehat{\mathbf{Q}}(t)$, $\mathbf{I}(t)$

for all firms i **do**

$\gamma_i(t+1) \leftarrow \texttt{production\_function}\left([Q^a_{ij}]_{j\in\partial^{in}_i}\right)$ ▷ Production begins

$I_{ii}(t) \leftarrow e^{-\sigma_i}\left(\mathscr{S}_i(t) - \sum_{j\in\partial^{out}_i} Q^e_{ji}\right)$ ▷ Firms update inventories for their own good

if $q = 0$ **then** ▷ If the economy is Leontief, firms need to stock other goods in addition to their own

$j^\star \leftarrow \arg\min_j \left([Q^p_{ij}]_{j\in\partial^{in}_i}\right)$

for all firms $j \in \partial^{in}_i$ **do**

$Q^u_{ij}(t) \leftarrow \frac{J_{ij}}{J_{ij^\star}} Q^p_{ij^\star}(t)$

$I_{ij}(t+1) = e^{-\sigma_j}\left[Q^a_{ij}(t) - Q^u_{ij}(t)\right]$

for all firms i **do**

$p_i(t+1) \leftarrow p_i(t+1)/p_0(t+1)$ ▷ Prices are updated

$B(t)$, $S(t+1)$, $p_0(t+1) \leftarrow B(t)/p_0(t+1)$, $S(t+1)/p_0(t+1)$, 1 ▷ Rescaling of monetary quantities

$C^d_i(t+1)$, $L^s(t+1) \leftarrow \texttt{compute\_demands\_labor}(S(t), L^s(t), L^d(t), \mathbf{p}(t+1), \omega', \varphi)$ ▷ The household starts planning

Output: $B(t)$, $S(t+1)$, $p_0(t+1) = 1$, $p_i(t+1)$, $\gamma_i(t+1)$, $C^d_i(t+1)$, $L^s(t+1)$, $\mathbf{I}(t+1)$