Booms, crises and socioeconomic phenomena

Petite Classe n. 4

Copycats and crashes

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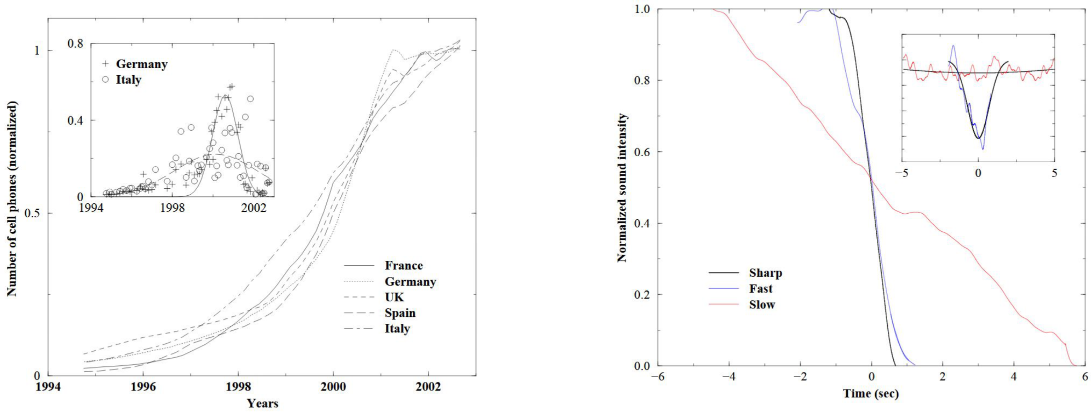
Introduction



(a) Cell-phone adoption in Europe.

(b) Time before the end of clapping in Radio France concerts

FIGURE 1: Two examples of collective phenomena (Q. Michard, J.P. Bouchaud, *Theory of collective opinion shifts : from smooth trends to abrupt swings*, 2005)

In this session we shall explore a simple, but very rich generic model : the Random Field Ising Model (**RFIM**). Although first used to model magnetic properties of disordered systems[1], it can be used to model binary-choice situations that can lead to non-trivial collective phenomena. As an example : to buy a cell-phone, to hold to or sell a stock...

Take a system of N, with each agent $i = 1 \dots N$ has a binary choice $S_i = \pm 1$, with the positive choice indicating for example that he wants to buy an asset. We suppose now that the agent wants to maximize an "utility function" given by :

$$U_i = h_i S_i + F S_i + S_i \sum_{j(\neq i)} J_{ij} S_j \tag{1}$$

where h_i is an idiosyncratic bias proper to each agent. We suppose that the h_i are random iid. variables with a density $\rho(h)$. The agent's decision is then naturally given by :

$$S_i = \text{sign}\left(h_i + F + \sum_{j(\neq i)} J_{ij} S_j \right) := \text{sign}(p_i) \tag{2}$$

where p_i is the "local polarization field".

1. Think of a metal with lots of impurities.

Part 1 : Preliminaries

1. Interpret the different terms in the equation. From now on, we take the mean-field limit $J_{ij} = J/N, \quad \forall(i,j)$. What does Eq. (2) look like now ?
2. What are the values of the S_i variables in the limits $F = \pm\infty$? How about when $J = F = 0$?
3. Consider now a fixed configuration of $(\{h_i\}, F, J)$, and consider the average opinion $m = \frac{\sum S_i}{N}$. How does this quantity vary when an agents changes their mind $S_i \to -S_i$?

Part 2 : Get your hands dirty

To begin, if you are not familiar with Python classes, read the example on the `Rectangle` class. For simplicity (see the next Part in the exercises) we will use a Laplace distribution for $\rho(h)$, namely

$$\rho(h) = e^{-|h|/\Delta}/(2\Delta) \tag{3}$$

1. Fill out everything on the class, except for the `equilibrate` method. You can also test things incrementally, but you are mostly independent here.
2. How can we equilibrate the system ? An equilibrium is a configuration of the S_i variables such that none can be "flipped" according to the decision rule of Eq. (2).
3. Create an instance of the `RFIM` with the value $J = 1$ and $\Delta = 0.1$. Start with $F = -10$ for the first one, and compute the equilibrium values $S_i(F = -10)$ and therefore the average opinion $m(F = -10) = \langle S_i \rangle$.
   Then increase the value of F progressively up to $F = 10$ (you can use `np.linspace(-10,10,100)` to do this in 100 points), and store the values of m you obtained for each value of F. (It's important you do this for the **same instance of the class**, changing F and reequilibrating succesively).
4. Do the same but going from $F = 10$ to $F = -10$. Plot the two curves you obtained for m. What do you notice ? Do the same again also but with $\Delta = 1.5$ and going from $F = -10$ to $F = 10$. Are there any differences ?

Part 3 : Solving the model

1. Suppose first that $\rho(h) = \delta(h)$. Starting from $F = -\infty$ and increasing F to ∞, when do all opinions change ?
2. Now take the Laplace density of Eq. (3) for ρ. Justify that as $N \to \infty$,

$$m = \langle \mathrm{sign}(h + F + Jm) \rangle_h \tag{4}$$

   where again $m = \frac{\sum S_i}{N}$. (hint : compute an average)
3. Considering that

$$\int \mathrm{d}h\ \rho(h)\mathrm{signe}\,(h + Jm(F) + F) = -1 + 2\int_{-\infty}^{Jm(F)+F} \mathrm{d}h\ \rho(h) := R(Jm(F) + F), \tag{5}$$

   show that R is an odd function with $\lim_{x\to\pm\infty} R(x) = \pm 1$, and that the slope of the tangent at the origin is $R'(0) = 2\rho(0) = 1/\Delta$, **you are not required to compute the function** R **explicitly**. Sketch the function.
4. Taking $J = 1$ for simplicity, show graphically that the equation $x = R(x + F)$ can have one solution for $\Delta > 1$ and one to three solutions for $\Delta < 1$.
5. Interpret the simulation in light of these results. Justify that there is a change in global opinion for values m^* and F^* given by :

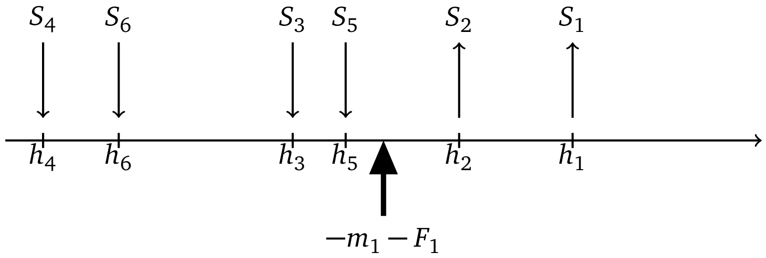
$$1 = 2\rho(m^* + F^*). \tag{6}$$

   The point (m^*, F^*) is called a critical value, analogue to the values of pressure and temperature in a phase transition, such as $(0°, 1013\text{hPa})$ et $(100°, 1013\text{hPa})$, that are the critical values for the solid/liquid and liquid/gas transition for water.

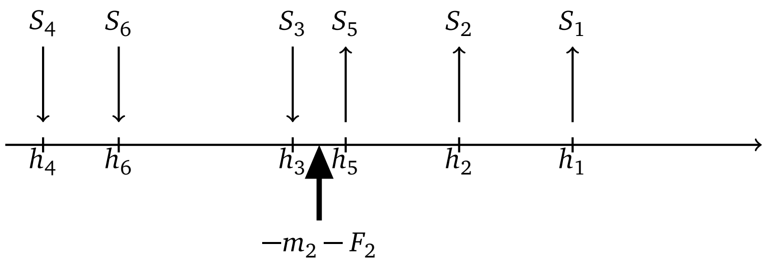
Part 4 : Dynamics (extra questions)

In the last part, we took directly the limit $N \to \infty$ to use the central limit theorem. In all that follows, we will consider the dynamics for a number $N \gg 1$ and look at the dynamics in the change of agents' opinions. We imagine that we begin with a value $F \ll 0$ that we increase "quasi-statically" (i.e. very slowly).

1. Here we zoom in the opinions of 6 individuals. The h_i variables are their intrinsic biases, and the black arrow shows the position of F. Graphically, $S_i = 1 :=\uparrow$ et $S_i = -1 :=\downarrow$.



   Interpret the Figure and explain why it's a stable configuration. Remember that at equilibrium, $S_i = \text{signe}(h_i + F + m)$, avec $m = \frac{1}{N}\sum_i S_i$.
2. If F increases very slowly, why can you say an avalanche is about to begin with the 5th agent ?
3. F increases to $F_2 > F_1$ and the system is now in the following situation :



   Why did S_5 flip ? By how much did m change with said flip ? What is the new value of $-m-F$?
4. Using a similar figure, what is the necessary condition for the avalanche to remain of size 1 ? (Meaning that **only** S_5 changed their mind).
5. What is the condition for the avalanche to be of size 2 ? (So that S_3 flips too).
6. Using the same reasoning (and extending into what is called a Poisson branching process), one can show that for a given value of m, the probability to trigger an opinion avalanche of size s when the exterior polarization field has a value F is given by

$$P(s) = \frac{(s\lambda)^{s-1}}{s!} e^{-s\lambda}, \quad \lambda := 2\rho(m+F) \tag{7}$$

   using the Stirling approximation formula, $s! \underset{s\to\infty}{\sim} \sqrt{2\pi s}\left(\frac{s}{e}\right)^s$, show that in the limit $s \to \infty$ one has $P(s) \simeq \frac{1}{\lambda}\frac{1}{\sqrt{2\pi s^3}} e^{s(1-\lambda+\log(\lambda))}$.
7. What happens at the critical point ? What's this distribution called ? Comment.