PHENOMENOLOGY AND MODELLING OF FINANCIAL MARKETS

Petite Classe n. 7

Large Latent Order Books: square-root impact of meta-orders

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For a long-time, people believed that the price impact of a meta-order of size Q was linear with Q. Many simple models (Kyle model or Santa Fe model for instance), would predict such a linear impact. However, empirical studies showed that this linear dependency was spurious and that a square-root impact was a better fit with reality. In this tutorial, we will model the latent order book with a so-called "reaction-diffusion" dynamics and retrieve the square root law.

The model

In this model, we assume that orders may be place for any value of the price therefore taking a continuous limit. We denote by $\rho_A(x,t)$ and $\rho_B(t)$ the latent volume densities in the Ask/Bid sides. These quantities evolves according to a set of rules

- Latent orders diffuse with diffusivity constant D;
- Latent orders are canceled with multiplicative rate ν;
- New intentions are deposited with additive rate λ;
- When a buy intention meets a sell intention they are instantaneously matched. This happens at rate κ and we implicitely assume that latent orders are revealed in the vicinity of the trade price p_t;
- The trade price p_t is conventionally defined through the equation $\rho_B(p_t,t) = \rho_A(p_t,t)$.

We finally define the reduced latent order book density by $\phi(x,t) = \rho_B(x,t) - \rho_A(x,t)$.

Part 1 : Diffusive orders in the absence of cancelations or depositions

In this first part, we provide a microscopic derivation for the diffusion part of the orders. We assume that the agents placing the orders revise their reservation price between time t and $t+\delta t$ according to

$$p_i \to p_i + f_t \beta_i + \eta_{i,t} \tag{1}$$

where both β_i and $\eta_{i,t}$ are random variable with densities P_β and P_η.

1. Interpret the different terms of the update. What can you say about the distributions of β and η?
2. Show that the density of latent orders $\rho(x,t)$ at price x at time t, follows the evoltuion

$$\rho(x,t+\delta t) = \int d\beta d\eta dx' P_\beta(\beta) P_\eta(\eta) \rho(x',t) \delta(x - x' - \beta f_t - \eta),$$

   with $\delta(\bullet)$ the dirac distribution.
3. Deduce from the previous equation

$$\rho(x,t+\delta t) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \partial_x^n \rho \sum_{k=0}^{n} \binom{n}{k} f_t^k \langle \beta^k \rangle \langle \eta^{n-k} \rangle.$$

   This equation is called Kramers-Moyal expansion.
4. We formally write $f_t = V_t \delta t$ and $\langle \eta^2 \rangle = 2D\delta t$. Truncating the above equation at order 2 (quadratic approximation), show that we get

$$\partial_t \rho = -V_t \partial_x \rho + D \partial_x^2 \rho.$$

5. Show that, with the appropriate change of the variable x, we get the *diffusion equation* with diffusion constant D

$$\partial_t \rho = D \partial_x^2 \rho. \tag{2}$$

6. Verify that

$$G(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right),$$

   is a solution to the above equation.

Part 2 : Cancelations, depositions and reactions

The previous simple microscopic model provides an explanation for the diffusive features of orders. We will add the possibility to place and cancel orders with rates λ and ν.

1. (a) Justify why the cancelation acts multiplicatively on $\rho_{B/A}$.

   (b) Deduce that (2) is modified as follows

$$\partial_t \rho_{B/A} = D \partial_x^2 \rho_{B/A} - \nu \rho_{B/A}. \tag{3}$$

   (c) Show that the solution to the previous equation is given by

$$G_\nu(x,t) = e^{-\nu t} G(x,t).$$

2. (a) Justify why the deposition rate acts additively on $\rho_{B/A}$. If the execution price is p_t, on average, are buy (resp. sell) orders place above or below p_t?

   (b) Deduce that (3) is modified as follows

$$\begin{aligned} \partial_t \rho_A &= D \partial_x^2 \rho_A - \nu \rho_A + \lambda \Theta(p_t - x) \\ \partial_t \rho_B &= D \partial_x^2 \rho_B - \nu \rho_B + \lambda \Theta(x - p_t), \end{aligned} \tag{4}$$

   with Θ the step function.
3. Finally, justify that the above equations are modified with the same term $-\kappa R_{AB}(x,t)$ when including the reaction phenomenon. What is the interpretation of R_{AB}.

4. Deduce, from all the above discussion, the following equation of ϕ

$$\partial_t \phi = D\partial_x^2 \phi - \nu\phi + \lambda \text{sign}(x - p_t). \tag{5}$$

Part 3 : Stationary order book and market-impact

In this section, we will briefly discuss the reaction-diffusion equation obtained above. We then take the *infinite memory limit* $\lambda, \nu \to 0$ while keeping the ratio $\mathscr{L} \sim \lambda\nu^{-1/2}$ constant.

1. Show that the stationary shape of the order book is given by

$$\phi^{st}(\xi) = -\frac{\lambda}{\nu}\text{sign}(\xi)\left[1 - e^{-|\xi|/\xi_c}\right]$$

   with $\xi = x - p_t$ and $\xi_c := \sqrt{D\nu^{-1}}$.

2. Draw the shape of the stationary order book and exhibit a quantity $\mathscr{L}$ below which the order book can be considered linear. Take the infinite memory limit. What is the shape of $\phi^{st}(\xi)$.

   Since meta-order of size Q are too big to be executed at once, they are usually broken down and placed within a time-span of T with a rate m_t such that

$$Q = \int_0^T m_s ds.$$

   We can model this as an extra source term $m_t\delta(x - p_t) \cdot \mathbb{1}_{[0,T]}$ in the reaction-diffusion equation. Using results from partial differential equations, one can show that the solution of this equation in the infinite memory limit reads

$$\phi(x,t) = [G(x,t) \star (\phi_0(x)\delta(t) + m_t\delta(x - p_t) \cdot \mathbb{1}_{[0,T]})](x,t),$$

   where $\star$ denotes the space-time convolution and $\phi_0(x) = \phi(x,0) := \phi^{st}(x)$. Computing this convolution, one can show that

$$\phi(x,t) = -\mathscr{L}x + \int_0^{t\wedge T} \frac{dsm_s}{\sqrt{4\pi D(t-s)}} \exp\left(-\frac{(x-p_s)^2}{4D(t-s)}\right)$$

3. We assume that the execution rate is constant ($m_t := m_0$) and we place ourselves at the end of the execution ($t = T$). Show that

$$p_T = \frac{m_0}{\mathscr{L}} \int_0^T \frac{ds}{\sqrt{4\pi D(T-s)}} \exp\left(-\frac{(p_T - p_s)^2}{\sqrt{4D(T-s)}}\right).$$

4. It is straightforward to check that $p_s = A\sqrt{Ds}$ is a solution.

   (a) Show that

$$A = \frac{m_0}{\mathscr{L}} \int_0^1 \frac{du}{\sqrt{2\pi(1-u)}} \exp\left(-\frac{A^2(1-\sqrt{u})}{4(1+\sqrt{u})}\right).$$

   (b) In the limit $m_0 \ll \mathscr{L}D$, derive $A \approx \frac{m_0}{\mathscr{L}D\sqrt{\pi}}$.

5. Explain why the impact of the metaorder is p_T, assuming $p_0 = 0$. Deduce that $I(Q) \sim \sqrt{Q}$.