PHENOMENOLOGY AND MODELLING OF FINANCIAL MARKETS

Petite Classe n. 5

Toy model of herding behaviors

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Introduction

In a series of experiments carried on harvester's ants, entomologists Deneubourg and Pasteels stumbled upon a curious phenomenon. Provided with two *a priori* identical food sources, ants tend to prefer one over the other but sometimes switch withot apparent reason. Ants display an asymmetrical behaviour in a symmetrical situation. In the realm of social sciences, we can realte this behavior to the phenomenon known as "herding". Herd instinct in finance is the phenomenon where investors follow what they perceive other investors are doing rather than their own analysis. Herd instinct has a history of starting large, unfounded market rallies and sell-offs that are often based on a lack of fundamental support to justify either. In its article *Ants, rationality and recruitment*, economist Alan Kirman proposes a simple model to explain ants' behaviors and by extension "herding".

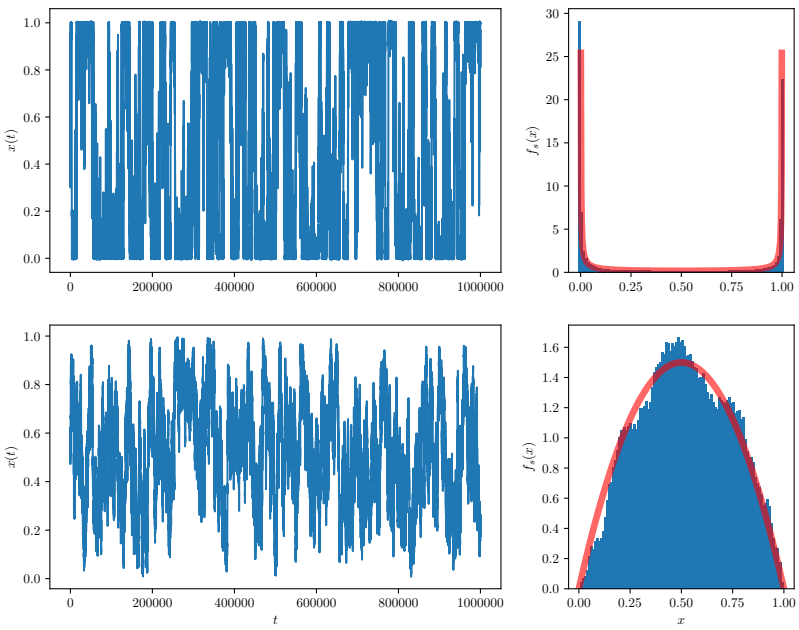

Figure 1: Top: Dynamics of $x(t)$, the proportion of ants in the first food source, along with the empirical and theoretical stationary distribution for $\alpha = 0.1$. Bottom: Same for $\alpha = 2$.

The model

Consider N ants divided between identical and always-full food sources denoting by F_1 and F_2. We denote by $k(t)$ the number of ants in zone F_1 at time t. Between time t and time $t+1$, we randomly

draw two ants (the order does not matter). The following things can happen

- The first ant can spontaneously switch to the other food source with probability $\epsilon \in]0,1]$. We exlude 0 to avoid all the ants being stuck in one food source.
- The first ant follows the second ant with probability $\mu \in [0,1]$.

We denote by $\mathbb{P}(k,t)$ the probability to find k ants in F_1 at time t; and by $W(\ell \to k;t) = \mathbb{P}(k(t+1) = k|k(t) = \ell)$, the transition rate between the state of ℓ to k ants in F_1.

Part 1 : Naive simulation

Simulate the dynamics in the simplest way possible and play around with the parameters ϵ and μ. What do you observe? Can you identify different regimes? Can you identify avalanches similar to the Random Field Ising Model?

Part 2 : Analysis of the model

1. Justify that the transition rates do not depend on the entire history of the system but only on the present time-step.
2. Let $k \in \{0,\dots,N\}$.
   (a) Show that, whenever $\ell \notin \{k-1,k,k+1\}$ then $W(\ell \to k) = W(k \to \ell) = 0$.
   (b) Show that the transition rates from k to $k+1$ and from k to $k-1$ read

$$W(k \to k+1) = \left(1 - \frac{k}{N}\right)\left(\varepsilon + \mu\frac{k}{N-1}\right) \tag{1}$$

$$W(k \to k-1) = \frac{k}{N}\left(\varepsilon + \mu\frac{N-k}{N-1}\right) \tag{2}$$

3. (a) Show that the probability to have k ants in F_1 at time t obeys the so-called Master Equation

$$\mathbb{P}(k,t+1) = \mathbb{P}(k-1,t)W(k-1 \to k) + \mathbb{P}(k+1,t)W(k+1 \to k) + \mathbb{P}(k,t)W(k \to k) \tag{3}$$

   (b) Denoting by $\mathrm{P}(t)$ the vector $\mathrm{P}(t)_k = \mathbb{P}(k,t)$, show that there exists a stochastic matrix T such that

$$\mathrm{P}(t+1) = \mathrm{T}^t\mathrm{P}(t)$$

   (c) Deduce the existence of a stationary probability measure P_s for the repartition of ants between the two food sources.

   In order to find the stationnary probability measure, we will take a continuous time and space limit as follows. In the master equation, we replace $t+1$ by $t+dt$ with $dt = \frac{1}{N} \ll 1$. We also replace μ by μN^2, ϵ by ϵN, and $\mathbb{P}(k,t)$ by the probability density function $f(x,t)$ with $x = k/N$. Finally, we take the limit $N \to \infty$.
4. Show that, with these replacements and new scalings, the master equation takes the form of a so-called Fokker-Planck equation in the limit $N \to \infty$

$$\partial_t f = \partial_x\left[-\epsilon(1-2x)f + \mu\partial_x\left[x(1-x)f(x,t)\right]\right]$$

5. Deduce that the stationary density function $f_s(x)$ is given by

$$f_s(x) = \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)}\left[x(1-x)\right]^{\alpha-1}, \tag{4}$$

   with $\alpha = \epsilon/\mu$ and $\Gamma(x) = \int_0^\infty dt\, t^{x-1}e^{-x}$. We give the identity $B(\alpha,\beta) := \int_0^1 x^{\beta-1}(1-x)^{\alpha-1}dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

6. Sketch this density and interpret it for different values of α.

Part 3 : Simulating the dynamics

Using the correspondence between a Fokker-Planck equation and a Langevin equation, one can show that the Fokker-Planck equation obtained in the previous section corresponds to the following dynamics for $x(t)$

$$\dot{x} = \epsilon(1-2x) + \sqrt{2\mu x(1-x)}\eta(t) \tag{5}$$

where η is a Gaussian white noise of unit variance. We can get the discreteized version of this dynamics with Itô rules

$$x_{t+dt} - x_t = \epsilon(1-2x_t)dt + \sqrt{2\mu x_t(1-x_t)}\sqrt{dt}\eta_t \tag{6}$$

with dt the step size. We will fix $\Delta t = 10^{-3}$.

1. (a) Write a function that simulates the above dynamics for given values of ϵ and μ.
   (b) Plot the fraction of ants in F_1 for $T = 10^5$ iterations, $\epsilon = 0.15$ and $\mu = 0.3$. What do you observe? What is the associated stationnary density?
   (c) Same question for $\epsilon = 0.002$ and $\mu = 0.01$.
2. (a) Plot the empirical stationnary distributions for the following values:
      - $\epsilon = 0.1$, $\mu = 0.01$,
      - $\epsilon = 0.1$, $\mu = 1$,
      - $\epsilon = 0.1$, $\mu = 0.3$.
   (b) Plot the associated theoretical stationnary distribution.