Phenomenology and modelling of financial markets

Petite Classe n. 7

Langevin dynamics

ECO 586 - T.Dessertaine
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Introduction

In this PC session we will study a simplified stochastic model on price fluctuations, with a clear physical idea in mind: a particle in a energy well subject to random fluctuations.

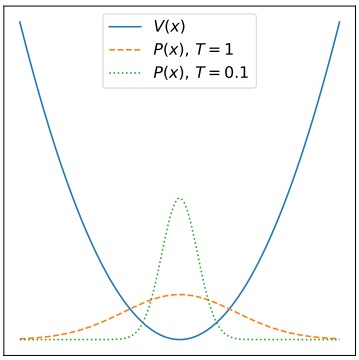

Figure 1: Example of the probability distribution of the position of a particle in a quadratic potential, $F(x) = -2x$.

The Langevin equation was proposed in 1908 as a way to model brownian motion. An interesting aspect of it is that it gives a clear physical interpretation to some stochastic differential equations. Consider then a particle of mass m in a viscous fluid with damping coefficient γ, and denote by x its position[1], then its dynamics are given by:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\gamma\frac{\mathrm{d}x}{\mathrm{d}t} + F(x) + \eta(t), \tag{1}$$

where F is a force field, $\eta(t)$ a random force (think of shocks by the fluid) with a characteristic time correlation $\tau_c \ll \mathrm{d}t$, modelled thus by $\langle\eta(t)\rangle = 0$ and $\langle\eta(t)\eta(t')\rangle\,\mathrm{d}t = 2T\delta(t-t')$.

In the over-damped limit $\frac{m}{\gamma} \ll 1$ the $\ddot{x}$ term is negligible, and if the force $F(x) = -V'(x)$ is conservative [2], the equation becomes:

$$\begin{aligned} \gamma\dot{x} &= F(x) + \eta(t) \\ \dot{x} &= -V'(x) + \eta(t), \end{aligned} \tag{2}$$

where we take $\gamma = 1$ for simplicity. This has the clear interpretation of a particle doing a "stochastic gradient descent" (going towards a local minimum of V but with fluctuations).

[1] We consider this 1-D case for simplicity. The generalization to 2, 3 or more dimensions is direct.

[2] Meaning that one can write it as minus the derivative of a potential, as done here.

One can show that the stationary probability distribution of the particle's position x is given by

$$p(x) \propto \exp\left(-\frac{V(x)}{T}\right), \tag{3}$$

that is the Boltzmann distribution for a temperature T. This simple example shall be used to develop your intuition on a simple model of returns with feedback.

An important thing to know is the Arrhenius law: the time needed to go over a potential barrier of height ΔV scales as

$$t^* \propto \exp\left(\frac{\Delta V}{T}\right). \tag{4}$$

Part 1 : A simple example: the Mexican hat potential

Consider first the potential $V(x) = \frac{a}{4}x^4 + \frac{b}{2}x^2$, with $a > 0$.

1. Study the potential V. How many minima are there for $b > 0$? For $b < 0$? Give the positions of the minima, their depth δ and sketch or plot the potential for $b > 0$ and $b < 0$.
2. From now on, we pick $a = |b|/2$ to fix the positions of the minima. We choose $T = 1$ where δ is the depth of the potential, and discretize the dynamics as:

$$x_{t+\mathrm{d}t} = x_t - V'(x_t)\mathrm{d}t + \sigma\xi_t\sqrt{\mathrm{d}t}, \tag{5}$$

   where the $\{\xi_t\}$ are gaussian iid variables.
3. Implement the dynamics to your liking[3] with $\sigma = 1$, and run them with $\mathrm{d}t = 10^{-3}$ for $5 \cdot 10^4$ timesteps for different values of b (e.g. $b \in \{-10, -0.5, 2\}$). Plot x_t for these different cases. What do you notice ? How can you explain this ?
4. Suppose we are in the situation $b < 0$. What happens if we observe the dynamics for a short (relative to b) timescale ? What is the name of this process ?

Part 2 : A model for returns: simplified case

We now consider returns r_t, but we consider that the price of the asset is fixed - i.e. that the returns are so small that they don't change the price on short enough timescales. We introduce the following model for their dynamics:

$$r_{t+\mathrm{d}t} - r_t = -\left(\kappa(p - p_F) + (a - a')r_t + \beta r_t^2\right)\mathrm{d}t + \sigma\xi_t\sqrt{\mathrm{d}t} \tag{6}$$

1. Interpret the different terms of this equation. From now on we redefine $\alpha = a - a'$. What is the corresponding potential ? What do you get when $\beta = 0$? What happens depending on the sign of α?
2. Pick $p_F = 0$ and $\kappa, p = 1$ and plot it for $(\alpha > 0, \beta > 0)$ and $(\alpha < 0, \beta > 0)$. What do you expect to see in the dynamics?
3. Do the same plots as previously but vary p from -1 to 1. What do you see ?
4. Implement the dynamics for $\alpha > 0$ and $\beta > 0$, but try $p = -1$ and $p = 1$. Comment on the observed results.

Part 3 : A model for returns: introducing feedback

We now introduce the fact that returns cause the price to vary, which in turn modifies the potential. The evolution is now given by:

$$\begin{aligned} r_{t+\mathrm{d}t} - r_t &= -\left(\kappa(p - p_F) + (a - a')r_t + \beta r_t^2\right)\mathrm{d}t + \sigma\xi_t\sqrt{\mathrm{d}t} \\ p_{t+\mathrm{d}t} - p_t &= r_t \cdot 10^{-2} \end{aligned} \tag{7}$$

[3] Two possibilities: write a function that takes a,b,dt,T and returns x_t for a run of T timesteps, or write a class with a self.timestep() method.

1. In light of the previous question, what do you think is going to happen ?
2. Implement the dynamics and simulate them for $(\alpha > 0, \beta > 0)$. Explain your results.
3. Same question for $\alpha < 0$ and $\beta > 0$.