# Phenomenology and modelling of financial markets

## Petite Classe n. 9

### Herd behavior and aggregate fluctuations in financial markets

ECO 586 - T. Dessertaine
M. Benzaquen

6 January 2020

In 1998, Rama Cont and Jean-Philippe Bouchaud proposed a simple model of returns that could account for the excess kurtosis of their distribution. As we empirically observe, returns (whether absolute or relative) are heavy-tailed and seem to be properly fitted by a student law. Furthermore, they display a shear assymetry - evidence of the difference of perception of positive or negative returns - along with non-trivial correlation structures. The model proposed here will only account for the heavy-tailed nature of returns (but we can imagine way to introduce skewness and long-range correlations) by introducing herding with a network structure.

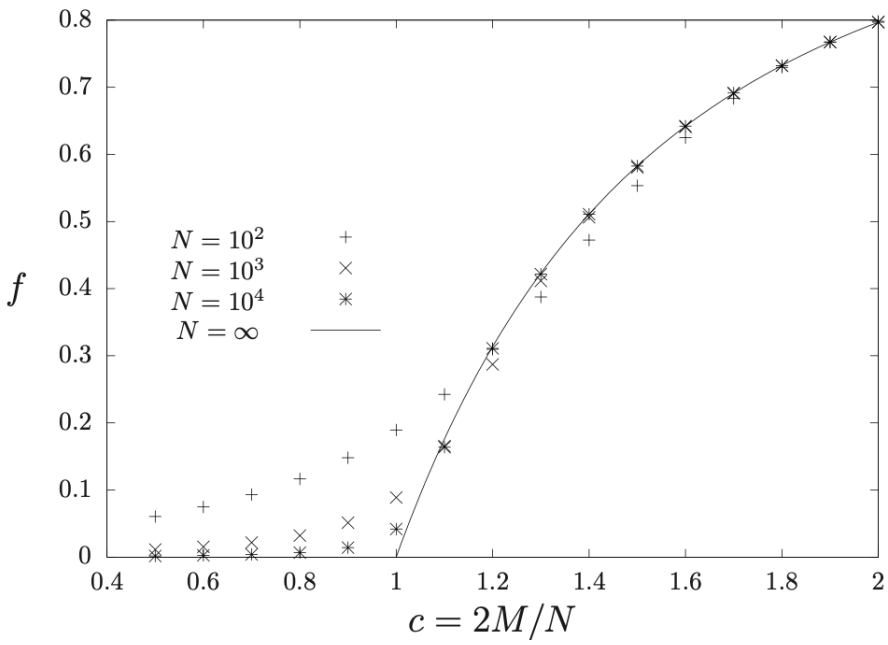


Figure 1: Percolation transition in Erdös-Renyi random graphs.

## The model

We consider $N$ agents trading a single asset. Each agent $i$ has a trading intention $\phi_i(t) \in \{-1, 0, 1\}$: $-1$ represents a sell intention, $1$ a buy and $0$ an agent doing nothing. At each time $t$, these intentions are chosen randomly with probability $a$ for $\pm 1$ and $1 - 2a$ for $0$. The underlying process of why an agent choses either one of the intentions is not specified.

Furthermore, we assume that agents display herding behavior: they cluster with each other and share a common trading intention within one cluster. The clustering procedure is as follows. Each eagent may form a link with the $N - 1$ others. A link is formed with probability $p$ independent of the

link. The average number of links around one agent is $p(N-1)$. For this number to remain finite in the limit $N \gg 1$, we choose $p = c/N$. Such procedure for constructing the graph has first been brought up by Erdös and Renyi. Interestingly enough, these graphs display a phase transition for the size of their largest connected component $S$. If $c > 1$ then $f := S/N > 0$, otherwise $f \to 0$ (see Fig 1). On can show that the size $W$ of one randomly chosen connected component of the graph is such that

$$\mathbb{P}(W = w) \underset{w\to\infty}{\sim} \frac{A}{|w|^{5/2}} \exp\left(-(c-1)\frac{w}{w_0}\right), \tag{1}$$

which becomes a pure power-law for $c = 1$.

Finally, as a first approximation, we assume that the price variation associated with the trading intentions $\{\phi_\alpha\}$ depends linearily on the intentions. As a result, we model the price change of the asset as follows

$$\Delta p(t) = p(t+1) - p(t) = \frac{1}{\lambda}\sum_{\alpha=1}^{n_c} W_\alpha \phi_\alpha(t) := \frac{1}{\lambda}\sum_{\alpha=1}^{n_c} X_\alpha(t), \tag{2}$$

with $n_c$ the (random) number of clusters, $W_\alpha$ the size of cluster $\alpha$ and $\phi_\alpha$ the associated trading intention. The parameter $\lambda$ is called *market depth* and measures how much trading intention is needed if one wants to move the price by one unit.

## Part 1 : Analysis of the model

1. Argue as to why the $X_\alpha$'s are independent identically distribted random variables. What is the distribution of their sum (when the sum is non random)?
2. Denoting by $F(x) = \mathbb{P}(X_\alpha \leq x | \phi_\alpha \neq 0)$, show that

$$\mathbb{P}(X_\alpha \leq x) = (1-2a)\Theta(x) + 2aF(x),$$

   with $\Theta(\bullet)$ the step function.
3. We assume that $F$ is associated to a density $f(x) := \mathbb{P}(X_\alpha = x | \phi_\alpha \neq 0)$. Explain why $f$ is even and why it displays the same asymptotic behavior as (1) when $|x| \to \infty$.
4. Show that the probability for a return to be equal to $\xi$ at time $t$ is given by

$$\mathbb{P}(\Delta p = \xi) = \sum_{k=1}^{N} \mathbb{P}(n_c = k) \sum_{j=0}^{k} \binom{k}{j} (2a)^j (1-2a)^{k-j} f^{\star j}(\lambda\xi)$$

   with $(\bullet)^{\star j}$ the $j$-fold convolution product.
5. We introduce the function $\tilde{f}(z) = \mathbb{E}[e^{X'_\alpha z}]$ and $\mathcal{F}(z) = \mathbb{E}[e^{\lambda \Delta p z}]$ (i.e the generating functions) where $X'_\alpha = \pm W_\alpha$. Show that

$$\mathcal{F}(z) = \sum_{k=1}^{N} \mathbb{P}(n_c = k)[1 - 2a + 2a\tilde{f}(z)]^k = \exp\psi\left(\log\left(1 - 2a + 2a\tilde{f}(z)\right)\right)$$

   with $\psi$ the cumulant generating function of $n_c$. We denote by $\gamma$ the quantity $1-2a+2a\tilde{f}(z))$.
6. Using graph theory, one can show that

$$\psi(z) = Nz + \frac{Nc}{2}(e^{-z} - 1).$$

   Deduce

$$\mathcal{F}(z) = \gamma^N \exp\left[\frac{Nc}{2}\left(\frac{1}{\gamma} - 1\right)\right].$$

7. What is the average number $n_o$ of trading agents in the market? How can it remain finite in the limit $N \to \infty$?

8. With this prescription, show that in the limit $N \to \infty$, one gets

$$\mathcal{F}(z) \approx \exp\left[n_o\left(1-\frac{c}{2}\right)(\tilde{f}(z)-1)\right].$$

9. We give the second moment $\mu_2(\alpha) = A(c)(1-c)^{-1}$ and the fourth moment $\mu_4(\alpha) = A(c)(1+2c)(1-c)^{-5}$ of $W_\alpha$. Deduce the kurtosis of the distribution of the price variation $\lambda\Delta p$

$$\kappa(\lambda\Delta p) = \frac{1+2c}{n_o\left(1-\frac{c}{2}\right)A(c)(1-c)^3}.$$

10. Comments?

## Part 2 : Simulations

Using the `networkx` package (imported as `nx`), one can easily generate random Erdös-Renyi graphs using

```
nx.fast_gnp_random_graph(N,p)
```

1. For $N = 10^4$, $n_o = 2000$, $\lambda = 10$, generate a series of returns on $T = 10^4$ and for $c = 1$.
2. Plot the histogram of positive and negative log-returns. Compare it to a Gaussian distribution. What is the exponent of the tail?

## Part 3 : Critical discussion

What are the main features of financial returns that are not reproduced by this simple model? Can you propose some mechanisms to recover these features?