Phenomenology and modelling of financial markets

Petite Classe n. 8

Portfolios

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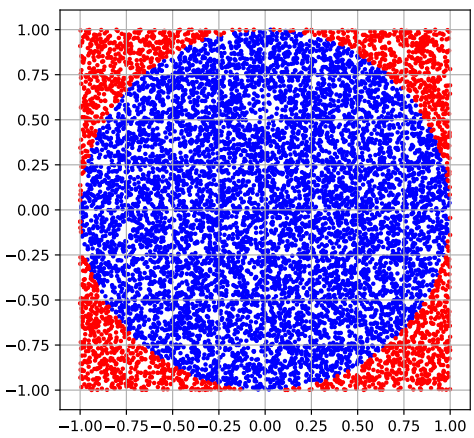
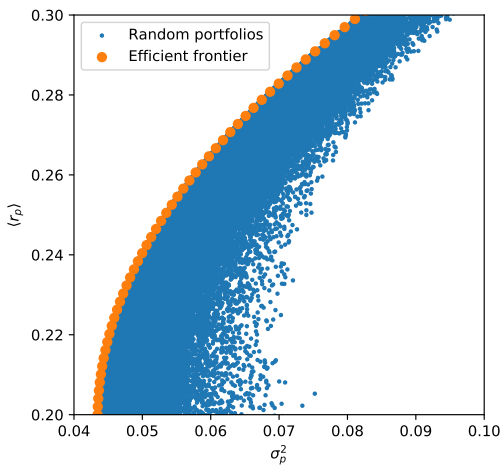
Part 1 : Monte-Carlo simulation for visualizing the efficient frontier

In this first exercise, we will use a Monte-Carlo simulation to visualize the efficient frontier for real assets. A Monte-Carlo simulation is a basic idea of what one calls "random sampling". The "historical" example is the estimation of the number π. In the sand of the Monte-Carlo beach, draw a square of side 2 along with the circle inscribed within this square. If you throw a pebble randomly within the square, the probability that it will land in the circle is $\pi/4$ (ratio of areas). If you have an infinite amount N of pebbles (and of time), you can throw them all within the square and count the number N_c of those landing inside the circle. The ratio N_c/N will converge towards $\pi/4$.



(a) Example of Monte Carlo simulation.



(b) Random portfolios to visualize the efficient frontier.

In the case of portfolios, we will consider 4 assets (Netflix, Apple, Facebook and Google). We will extract the annual empirical covariance matrix of these assets along with their mean returns. We will use these information to construct random portfolios.

1. (a) In the following, we will consider the portfolios to be "fully-invested" i.e $\sum_i w_i = 1$. Show that the Markowitz solution for risk optimal portfolios on N assets with return target $\mathcal{G}$ (correlated through the matrix $\mathbf{C}$ and with mean returns $\boldsymbol{\mu}$) is given in this case by

$$\mathbf{w}^\star = \frac{1}{2}\mathbf{C}^{-1}\left[\lambda\boldsymbol{\mu} + \gamma\mathbf{1}\right], \tag{1}$$

with $\mathbf{1} = (1, \ldots, 1)^\top$ and λ, γ solution of

$$\frac{1}{2}\begin{pmatrix} \boldsymbol{\mu}^\top\mathbf{C}^{-1}\boldsymbol{\mu} & \boldsymbol{\mu}^\top\mathbf{C}^{-1}\mathbf{1} \\ \mathbf{1}^\top\mathbf{C}^{-1}\boldsymbol{\mu} & \mathbf{1}^\top\mathbf{C}^{-1}\mathbf{1} \end{pmatrix}\begin{pmatrix} \lambda \\ \gamma \end{pmatrix} = \begin{pmatrix} \mathcal{G} \\ 1 \end{pmatrix}. \tag{2}$$

(b) What is the solution for (λ, γ)? Does it always exist?

2. Define a function `optimalPortfolio` that takes $\mathbf{C}$, $\boldsymbol{\mu}$ and $\mathcal{G}$ as arguments and builds a risk-optimal fully-invested portfolio with average return $\mathcal{G}$.
3. Use the given code to retrieve the 'Adj Close' value for the assets Netflix, Apple, Facebook and Google. Store it in a data-frame `dfRaw`.
4. Extract the vector of daily mean-returns `dailyReturns` and deduce the annual mean-returns `annualReturns`. Same for the covariance matrix (`dailyCov` and `annualCov`).
5. Define a function `genRandomPortfolio` that takes $\mathbf{C}$, $\boldsymbol{\mu}$ as arguments and returns the wieghts, the risk σ_p^2 and the mean return $\langle r_p \rangle$ of a random fully-invested portolio.
6. Using the previous function, generate 10^5 portfolios. Store their risk and mean returns and plot them in the plane $(\sigma_p^2, \langle r_p \rangle)$. What do you observe?
7. From the range of average returns that you observe, plot the efficient frontier using the function defined in 2.

Part 2 : Long-only portfolios and spin-glass theory

In this section, we will consider the problem of the sparsity of large long-only portfolios, using a parallel with spin-glass theory in statistical physics. Long-only portfolios (and more generally trading strategies) are characterized by $w_i > 0$ for all i. We will consider N assets correlated through a matrix $\mathbf{C}$ that takes the following form, known as the single-index risk model

$$\mathbf{C} = \mathbf{I}_N + \boldsymbol{\beta}\boldsymbol{\beta}^\top, \tag{3}$$

where β_i is the asset i's volatility relative to the market. We assume that these assets have a mean-return $\boldsymbol{\mu} = \mathbf{1}$ and that the target mean-return of our portfolio is $\langle r_p \rangle = 1$. In this case, one can write the Markowitz solution as

$$w_i \propto \sum_j C_{ij}^{-1}, \tag{4}$$

up to a normalizing constant. We give the so-called Sherman-Morrison formula for $\mathbf{A}$ a non-singular matrix and $\mathbf{u}, \mathbf{v}$ two vectors such that $1 + \mathbf{v}^\top \mathbf{A}^{-1} \mathbf{u} \neq 0$

$$(\mathbf{A} + \mathbf{u}\mathbf{v}^\top)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^\top\mathbf{A}^{-1}}{1 + \mathbf{v}^\top\mathbf{A}^{-1}\mathbf{u}}. \tag{5}$$

1. What does a long-strategy refer to? A short?
2. Interpret the single-index risk model.
3. Show that the Markowitz solution for the single-index risk model is given by

$$w_i \propto 1 - \beta_i \frac{\sum_j \beta_j}{1 + \sum_j \beta_j^2}. \tag{6}$$

A natural question that may arise is: how many long-only portfolios can be build according to (6)? We can turn to statistical physics and more precisely to spin-glass theory to try to answer this question. We introduce a "spin" θ_i associated to w_i that takes the value 1 if $w_i > 0$ and 0 if $w_i < 0$. Furthremore, we randomize the value of β to allow a statistical treatment. In the following, we will consider that $\boldsymbol{\beta} \hookrightarrow \mathcal{N}(\mathbf{1}, \sigma^2 \mathbf{I}_N)$.

4. (a) Write a function `marSirm` that generates a fully-invested Markowitz portfolio of the form (6) for a given vector β. Generate one portfolio. Is it long-only?

   (b) To construct a long-only Markowitz portfolio of the form (6), we use the following procedure:

   - Generate a portfolio w_0 using `marSirm`. Compute the associated configuration θ_0. Recompute a portfolio w_1 excluding positions i for which $\theta_{0,i} = 0$. Compute the associated configuration θ_1.
   - While $\theta_{i-1} \neq \theta_i$, repeat the procedure above.

   Code this procedure. (Hint: you can modify the function `marSirm` to include θ.)

5. We introduce a variable

$$m = \left\langle \frac{1}{N} \sum_i \theta_i \right\rangle_\beta \tag{7}$$

with once again $\theta_i = 1$ if $w_i > 0$ and 0 otherwise. Interpret this quantity.

6. We will consider the parameter $\chi = \sigma N$.
   (a) Write a function that generates a long-only portfolio for a random draw of the vector β with σ and χ fixed.
   (b) For $\sigma \in \{0.1, 0.01, 0.05\}$, plot the average value of m as a function of χ. What do you notice?
   (c) Interpret the previous behavior in term on long-only investment strategies.